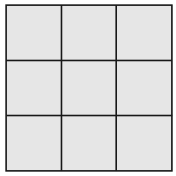
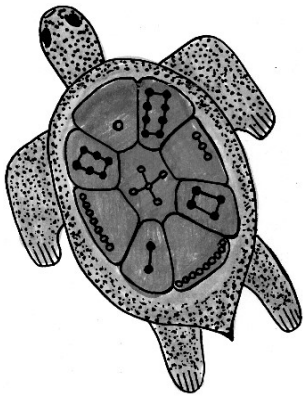
Chapter 5 Magic Square Puzzles

Never be afraid to sit awhile and think.
–Lorraine Hansberry

5.1 Introduction to Magic Squares

MS Puzzle 1 Place the numbers 1 through 9 in the nine squares of the grid at right. Use each number exactly once and arrange them so the sum of the numbers in each row, column, and both main diagonals is the same. This puzzle is called a magic square.





Most math historians believe the magic square originated in China around 2200 BCE, where it was associated with the Legend of Lo-Shu. According to the legend, there was a huge flood on the river Lo and in order to calm the river god's anger, the peasants of the river valley left sacrifices. However, after each sacrifice a turtle appeared and walked around the offerings but did not take them. The people interpreted this rejection of the offerings to mean that the river god did not accept the amount of the sacrifice. After this happened a number of times, a child noticed markings on the turtle's shell. The peasants realized the sum of the markings in each row, column, and both main diagonals was 15 and thus 15 should be the number of offerings. The puzzle was solved! Over the centuries this magic square appears as good luck charms called Lo-Shu. The 3×3 magic square is found in cultures all over the world including the ancient Babylonians, the ancient Greeks, and the ancient Mayans. In Islamic cultures the magic square symbolized the power of Allah.

What is a Magic Square?

A **magic square** is a square array of distinct integers such that the numbers in any row, column, or main diagonal have the same sum (called the magic sum or magic constant). If the n^2 numbers in an n by n magic square are the positive integers 1 through n^2, then the magic square is a **normal magic square**. The 3×3 magic square above, when completed using the numbers 1 through 9, is a normal magic square. To calculate the magic sum for any normal $n \times n$ magic square, see Appendix 7.

3×3 Magic Square Sculpture
Artist: Brian O'Doherty/Patrick Ireland

The smallest magic square (1×1) is of little interest since there is only one square with one number in it. Can there be a 2×2 magic square? This would be an arrangement of four consecutive positive integers 1, 2, 3, and 4, so the sum is the same in each row, column, and both main diagonals. Once a number is placed in one of the four squares, it becomes impossible to place any other number to create a magic square.

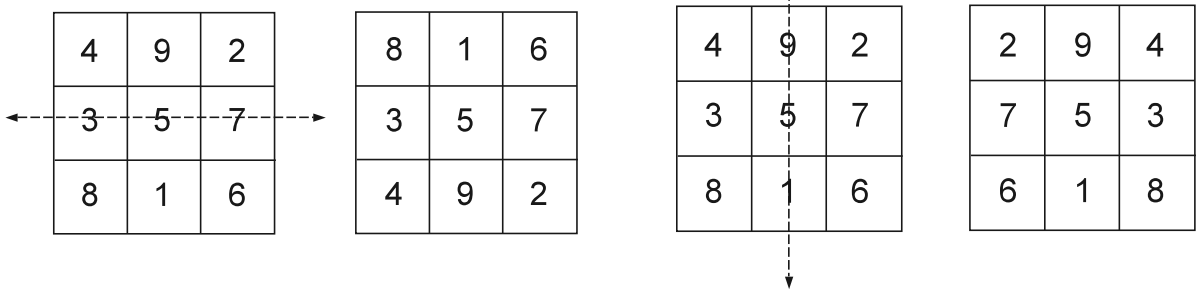
1	?
?	?

5.2 3x3 Magic Squares

If you were able to solve the puzzle from the previous page, your solution may have looked like the one shown at right.

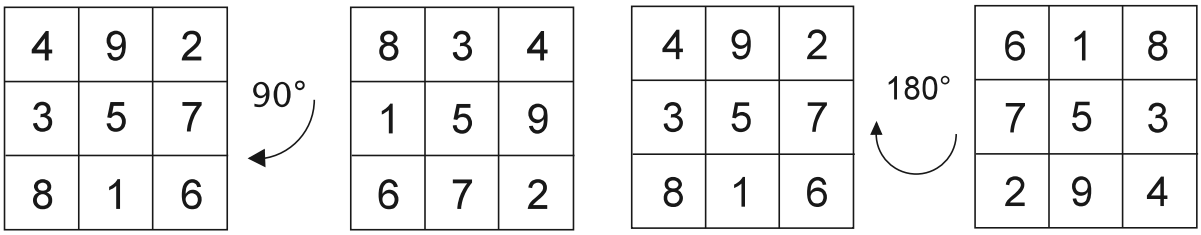
4	9	2
3	5	7
8	1	6

Are there other solutions? Well, if we reflect it horizontally or vertically it is still the same magic square but with a different orientation.



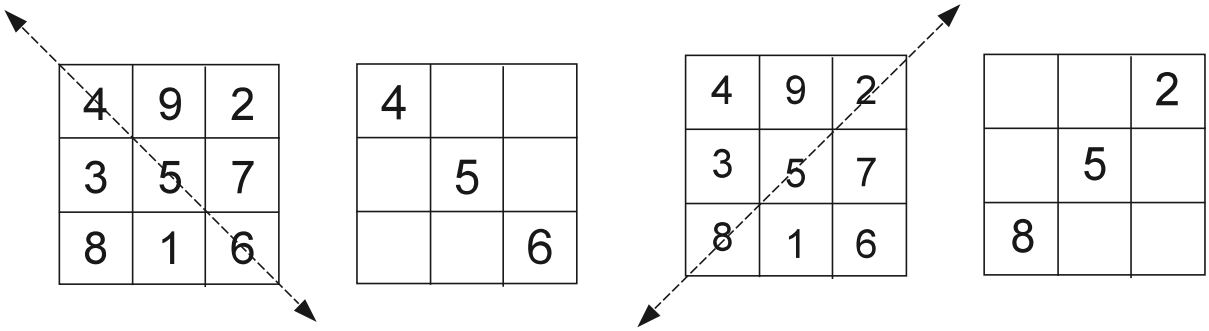
Reflected top to bottom **Reflected right to left**

If we rotate the magic square 90° or 180° is it still the same magic square?



Rotated 90° **Rotated 180°**

If we reflect the original magic square across a main diagonal, is it still the same magic square? Try it. Complete each grid below by reflecting the remaining digits across a main diagonal.



Reflected across the 4-5-6 diagonal **Reflected across the 2-5-8 diagonal**

You've seen seven different reflections and rotations of the same 3×3 magic square. There is one more. Can you find it? See Appendix 7 for all eight arrangements.

4	9	2
3	5	7
8	1	6

Is there a different 3×3 magic square that is not just a rotation or reflection of the original? If we cannot find one, can we prove that one does not exist? Go to Appendix 7 for the proof.

In the exercises below, take the basic normal 3×3 magic square shown at right and perform the given arithmetic operation on each number.

Add 5 to each

Subtract 4 from each

Multiply each by 2

Check that these squares are also magic squares. Magic squares that do not use just the first nine positive integers, such as those above, are called **simple** 3×3 magic squares. This definition extends to any $n \times n$ magic square that does not use just the first n^2 positive integers.

An incomplete magic square, like the grids below, is a magic square puzzle in which your task is to figure out the missing numbers. Complete the 3×3 simple magic square puzzles below. Place the given numbers so that the sum of the numbers is the same in each row, column, and on both main diagonals.

MS Puzzle 2 *h*

12	7	
5		13
		6

Use 8,9,10,11

MS Puzzle 3

	14	
18		2
	6	16

Use 4,8,10,12

MS Puzzle 4

11		15
3		7

Use 1,5,9,13,17

MS Puzzle 5

	8	
2		26
	20	

Use 5,11,14,17,23

MS Puzzle 6

	11	
7		-1
	-5	

Use -3,1,3,5,9

MS Puzzle 7

19		
		22
	16	

Use -2,1,4,7,10,13

Solve these puzzles using the given clues.

MS Puzzle 8 *h*

9		
	11	
	15	

The magic sum is 33

MS Puzzle 9

13		
20		12

The numbers are consecutive

MS Puzzle 10

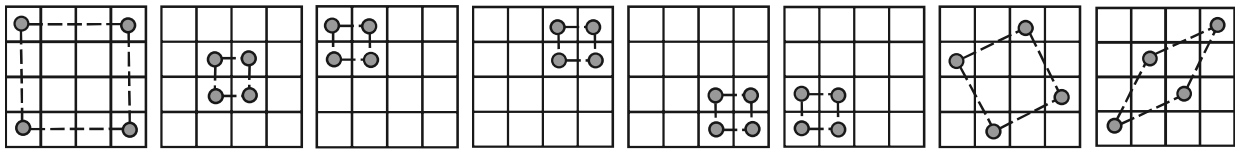
	50	
56		52

All consecutive even

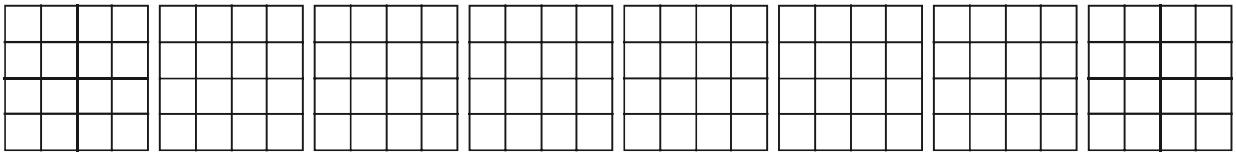
5.3 4x4 Magic Squares

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The magic square shown at right is a normal 4×4 magic square. Unlike the normal 3×3 magic square, which has only one solution, there are 880 normal 4×4 magic squares. This does not even count the reflections and rotations of each of the solutions! This 4×4 magic square is a favorite of many mathematicians. It appeared in Renaissance artist Albrecht Durer's woodcut titled, *Melencolia I.* A quick check reveals the magic sum for a normal 4×4 magic square is 34. To determine the magic sum for any $n \times n$ normal magic square, see Appendix 7. This magic square is even more magical than most; not only does every row, column, and both main diagonals add to the same magic sum of 34, but the magic sum also appears in more than two-dozen symmetric patterns. Each set of four squares, shown in the patterns below, has numbers that add up to the magic sum of 34.



MS Puzzle 11 Eight of the more than 24 symmetric patterns are shown in the previous diagrams. Find at least eight more patterns that sum to 34.



Complete each puzzle below. Puzzles 12–20 are all normal 4×4 squares, so they all have the same magic sum of 34.

MS Puzzle 12 *h*

	8	12	
3			15
2	11		
	5	9	4

MS Puzzle 13

7	2		
		3	6
1	8		15
	11	5	

MS Puzzle 14

13		2	16
8			
	6	7	
1			4

MS Puzzle 15

10		8	
5			14
	6	13	
	9	2	

MS Puzzle 16

	14		1
9			12
5		10	
	2	3	13

MS Puzzle 17

2	14	11	
1	13	8	
			9

MS Puzzle 18

	14		
5			
9		6	
	2	3	13

MS Puzzle 19

8		5	
	3		2
12	6		
	15		14

MS Puzzle 20

	2		
13		4	
	7		6
	11	5	

Puzzles 21-26 are simple magic squares. Solve them using the given integers.

MS Puzzle 21

	4		43
31		16	
19		28	10
	40		7

{-2,1,13,22,25,34,37}

MS Puzzle 22

25			1
		11	
	21	13	
31			7

{odd integers 1 thru 31}

MS Puzzle 23

19			
	10	9	
	14	13	
7			

{4 thru 19}

MS Puzzle 24

10			13
	7	8	
-2			1

{-2 thru 13}

MS Puzzle 25

	9	8	
14		17	
			15
	21	20	

{7 thru 22}

MS Puzzle 26

	11		
5		3	
	6	7	4
		-2	

{-3 thru 12}

Solve these puzzles using the given clues.

MS Puzzle 27 *h*

	17		18
		23	
			14
8		11	

The magic sum is 62.

MS Puzzle 28

		8	22
	16		
18	12		
7			

Use consecutive integers.

MS Puzzle 29

32	6		
10			16
			24
	30	28	2

All consecutive even.

5.4 5x5 Magic Squares

24	19	1	3	18
6	10	17	12	20
5	15	13	11	21
22	14	9	16	4
8	7	25	23	2

There are 275,305,224 normal 5×5 magic squares (again, not counting rotations and reflections). Here is an interesting example: the magic square at left is called a **bordered magic square**. Ignore or remove the outer border of squares of this 5×5 magic square and you get a 3×3 grid of numbers. Is there anything special about this 3×3 grid? You guessed it—it is a simple 3×3 magic square!

Notice that each number in the center 3×3 grid is eight more than the corresponding number in a normal 3×3. So the center 3×3 magic square has a magic sum of 39. The magic sum for a normal 5×5 is 65. Therefore, each pair of opposite numbers in the ring must add to 26. Are there other arrangements of numbers around the perimeter that give another 5×5 bordered magic square? Rotating the ring by 90°, 180°, or 270° produces three others. But there are still more which we will see in the next puzzle.

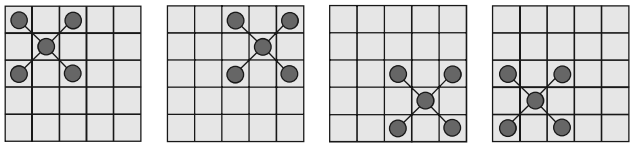
1	22			
	10	17	12	24
	15	13	11	
21	14	9	16	
			7	

MS Puzzle 30 Complete the 5×5 bordered magic square at right.

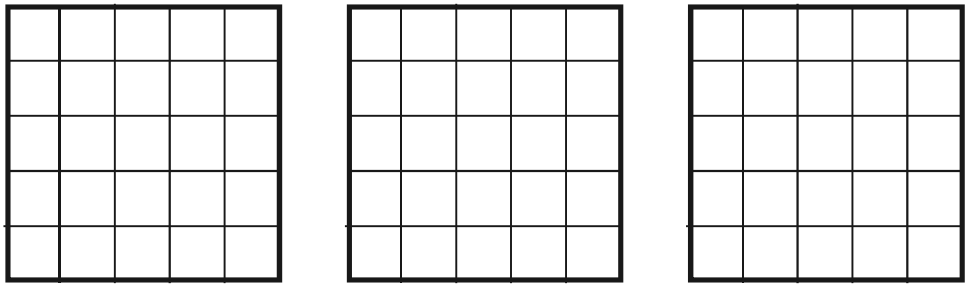
1	19	7	25	13
22	15	3	16	9
18	6	24	12	5
14	2	20	8	21
10	23	11	4	17

As you may recall, Albrecht Durer's 4×4 magic square contained more than two dozen symmetric patterns of connecting numbers whose magic sum is 34.

The magic square at left has a magic sum of 65. You will also find a number of symmetric patterns of connecting numbers whose sum are 65. Four of these patterns are shown in the diagrams below.



MS Puzzle 31 Find at least three more symmetric patterns that sum to 65.



Complete each 5×5 normal magic square puzzle below.

MS Puzzle 32

	24	1		
23	5		14	16
4		13		22
10	12			3
11		25	2	

MS Puzzle 33

	11	18		16
8		4	12	
				2
	6	23	1	10
14	9		24	

MS Puzzle 34

		4		14
11	3		19	9
		5	7	6
1			15	
	2	13		

MS Puzzle 35

	21		7	
	6			1
16		10		12
	8	11	22	
18	17	4		24

MS Puzzle 36

15				
		18		20
	23	21		12
24	14		16	10
6	8	22		4

MS Puzzle 37

	11	18		16
8		4	12	
				2
	6	23	1	10
14	9		24	

MS Puzzle 38

23			2	
10	18			22
	5		21	9
4			8	
	24	7		3

MS Puzzle 39

23				15
		25	8	
10	18	1		22
		7	20	
17			21	9

MS Puzzle 40

		13	2	
	15			1
6	7	5		
9	19		3	11
14		4		

Solve these 5×5 simple magic square puzzles using the given clues.

MS Puzzle 41 *h*

8	16		12	25
	22			1
20		11		
21				13
2	15	23		19

The magic sum is 65.

MS Puzzle 42 *h*

	29	16		25
31	23	40		
	17		21	38
		28	20	
18	35			26

Use consecutive integers.

MS Puzzle 43

		14		6
34		26		
46	12			30
	24	50		32
20		2	28	

All consecutive even.

5.5 6×6 and 7×7 Magic Squares

Solve the following 6×6 and 7×7 magic square puzzles.

MS Puzzle 44

6		3		35	1
7	11		28	8	
19	14	16			24
			21	17	
25	29		9		
	5	33	4	2	31

MS Puzzle 45

22		31	6		20	
40		10	9	44		15
7	1	30	49		32	35
		25				34
23	38			3	26	37
36	18	28	45	19		16
	29	5	17	41		27

5.6 8×8 Magic Squares

There are 275,305,224 normal 5×5 magic squares. There are an *estimated* 1.7745×10^{19} normal 6×6 magic squares. The number of $n\times n$ normal magic squares for $n > 5$, such as the 8×8 magic squares shown below, is still unknown. Solve the following 8×8 normal magic square puzzles.

MS Puzzle 46

43	47	2	26	1		37	59
54	18	63	17	14		58	
40	36		44	50	21	9	25
					32	38	51
	23	52	15		46	11	62
28	34		49		64	13	8
	19	5	41		33	55	29
31		57	56	7	3		

MS Puzzle 47

	17	52	36	19	48	59	4
43	33		47	38	57		16
49		24		11		10	28
1	60	46		64	12	35	
37		30		45		51	62
22	32		23	6	13	56	58
41	18	9	40			26	
	61	44	15	14	27		55