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FOR those who teach mathematics, using technology in the mathematics classroom has meant the increased use of Geometer's Sketchpad, spreadsheets, and graphing calculators. Although these technology applications are most often used in middle school and high school mathematics, many technology tools are appropriate for younger learners. An exciting technology for use in the early grades is called *virtual manipulatives.* Three elements have converged to form this class of manipulative technology. Those elements include innovations in computer technology that allow programmers to create electronic objects, the availability of Internet resources, and educators' recognition of the usefulness of concrete manipulatives and other representations for teaching mathematics. As a result of these advances, computer programmers created *virtual manipulatives.*

A virtual manipulative is defined as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard, and Spikell 2002, p. 373). Virtual manipulatives are often exact visual replicas of concrete manipulatives (such as pattern blocks, base-ten blocks, geometric solids, Cuisenaire rods, or geoboards). They are placed on the Internet as applets—smaller, stand-alone versions of application programs. Children can use the computer mouse to manipulate the images.

Teachers and researchers are discovering the useful properties of virtual manipulatives (Moyer and Bolyard 2002) and measuring their impact on stu-

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Editor's note: The CD accompanying this yearbook contains an electronic version of this article, complete with hyperlinked URLs.

dents' learning (Reimer and Moyer 2003). This paper discusses virtual manipulatives as a unique technological form of representation for teaching mathematics and describes how two elementary school teachers conducted action research projects in their classrooms using this technology. During these projects, a kindergarten teacher and a second-grade teacher taught a series of lessons and documented children's use of two virtual manipulatives (virtual pattern blocks and virtual base-ten blocks). In the process, the teachers gathered data on the children's use of technology and examined how it enhanced the children's understanding of mathematical concepts and skills.

Virtual Manipulatives: A Unique Technological Representation

In the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM 2000), virtual manipulatives fall under the Process Standard *Representation*. Although Representation is new as a process strand to the NCTM *Principles and Standards,* educators have used different forms of representation in teaching mathematics for many years. The most common forms of representation used in school mathematics include concrete/physical representations (manipulatives and 3D geometric models), pictorial/visual representations (pictures, drawings, and other visual images), and abstract/symbolic representations (numbers, letters, and operation signs).

In addition to these familiar representations used in mathematics, forms of representation have begun to appear on the Web. Some of these representations are static and are essentially pictures that can be viewed but not manipulated. (For example, a picture of pattern blocks with questions printed below the blocks would be a static representation.) These might be called "virtual math activities" because they appear on the Web, but they would *not* fit the definition of virtual manipulatives.

Other representations on the Web are dynamic, visual objects. Many of them look like concrete manipulatives and can be moved as one would move an object. The ability to move, or manipulate, these visual objects makes them interactive for children and makes them "virtual manipulatives." Children who interact with this different form of technology will have *different* mathematical experiences. As the Technology Principle indicates, "Work with virtual manipulatives … can allow young children to extend physical experience and to develop an initial understanding of sophisticated ideas like the use of algorithms" (NCTM 2000, pp. 26–27).

Because virtual manipulatives are a unique mathematics tool, one might wonder if virtual manipulatives are an additional form of representation. If we consider them in relation to the three forms of representation currently used in school mathematics (physical, pictorial, symbolic), virtual manipulatives do not fit neatly into any one of these three categories. Virtual manipulatives do furnish a visual image like a pictorial model, and yet they can be moved and manipulated like a physical model. In essence, they have some of the advantageous properties of both of these forms of representation, as well as some additional advantages brought about by their technological properties. Because of these attributes, virtual, concrete, and pictorial representations support one another in the mathematics teaching and learning process. Each representational form contributes important information that supports a more complete understanding of underlying mathematical ideas. Therefore, one should not be a substitute for another.

Virtual manipulatives are uniquely suited for teaching mathematics with young children. A Web connection makes them free of charge and easily *available.* Some virtual manipulatives have the *potential for alteration.* For example, users can color parts of objects; they can mark the sides of a polygon or highlight the faces on a Platonic solid. This *interactivity* allows all children to be engaged in the problem-solving process. Some virtual manipulatives *link symbolic and iconic notations* by saving numerical information or providing mathematics notations that label the on-screen objects. The click of the mouse on many virtual manipulatives gives children access to *unlimited materials.* And *clean up is easy* — children simply click an icon and the on-screen objects disappear.

Numerous virtual manipulative Web sites are currently available. Some of the largest collections of virtual manipulatives can be found at the NCTM Web site (www.nctm.org/), the National Library of Virtual Manipulatives (www.matti.usu.edu/nlvm), and the Shodor Education Foundation (www.shodor.org/).

Teaching Mathematics with Virtual Manipulatives

The classroom projects in this paper were conducted at Westlawn Elementary School in Falls Church, Virginia—a Title I school that serves a linguistically and economically diverse population. Just thirty minutes from the nation's capital, the school demographics include a population of students 75 percent nonwhite with 34 percent second-language learners. In each of the classes in the following vignettes, the children had some prior experiences with technology and with the virtual manipulative Web sites during mathematics lessons. Therefore, they were able to use the virtual manipulatives efficiently for the mathematical tasks the teachers designed.

Two of the authors of this article are classroom teachers, and they led instruction during each of the class sessions. Five observers recorded anecdotal notes on the children's work during the mathematics interactions. Each observer was assigned to a small number of children (three or four) so that the observer could record the children's work and ask questions throughout the lessons. Children's work was analyzed and condensed for this article to highlight major points in each of the lessons.

Kindergarten Patterns

The full-day kindergarten class was composed of eighteen children, twelve of whom spoke a language other than English in their homes. The group was also ethnically diverse, including Hispanic, Asian, African-American, Middle Eastern, and Caucasian students. During the three-day lesson on patterns, the children focused on both repeating and growing patterns of many kinds. For example, the children had previous experiences with simple repeating patterns (i.e., ABB, AAB, or ABAB) and had been introduced to growing patterns (i.e., A, AB, ABC, ABCD, or 1, 1-2, 1-2-3, 1-2-3-4). Children had explored patterns visually, aurally, and kinesthetically using shapes, colors, sounds, and rhythms. They also investigated patterns in literature, music, and nature. The ability to recognize, describe, and extend patterns is an essential mathematics Standard for grades K–2 (NCTM 2000).

On the first day of their three-day lesson on patterns, the children used wooden pattern blocks to make their patterns and a long strip of oaktag paper as a work mat. On the second day, the children used virtual pattern blocks to create their patterns. Virtual pattern blocks are available on several Web sites, including NCTM (nctm.org), Arcytech (www.arcytech.org/), and the National Library of Virtual Manipulatives (www.matti.usu.edu/nlvm). (Fig. 2.1 shows a child's pattern on the Arcytech Web site.) On the third day, the children created and drew patterns freehand on 4″ × 18″ strips of construction paper using crayons or markers.

To analyze the children's work, the project team examined how the form of representation (wooden pattern blocks, virtual pattern blocks, or drawings) influenced the variety and complexity of the children's patterns. The results that follow are summarized in table 2.1.

The team first compared the number of individual patterns each child constructed on each day. This comparison showed that the children made a greater number of patterns using the virtual pattern blocks than they did drawing the patterns or using the wooden pattern blocks. (See table 2.1.) An analysis of the variety of pattern types the children made each day indicated that the most common patterns were AB, ABB, AAB, ABC, and ABCD across all

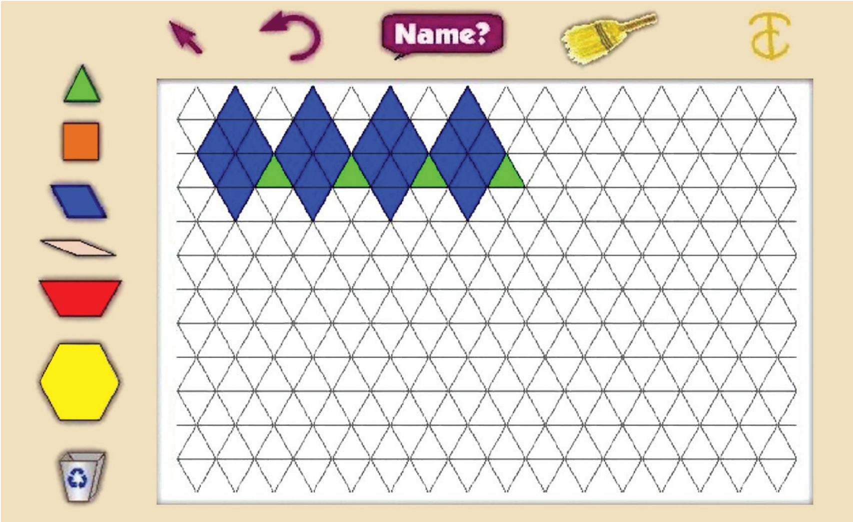

Fig. 2.1

three forms of representation. This analysis also indicated that the pattern types children made each day remained consistent. For example, those who created ABC patterns on Day 1 also made ABC patterns on Days 2 and 3.

Table 2.1
Quantitative Summary of Pattern Results — Kindergarten

Pattern Characteristics	Day 1 Wooden Pattern Blocks	Day 2 Virtual Pattern Blocks	Day 3 Children's Drawings
Total # of Patterns Created	48 pat/18 st Avg. = 2.7 pat/st	63 pat/18 st Avg. = 3.5 pat/st	46 pat/17 st Avg. = 2.7 pat/st
# of Blocks in Each Pattern Stem	51.33 bl/18 st Avg. = 2.9 bl/stm	59.75 bl/18 st Avg. = 3.3 bl/stm	48.17 bl/17 st Avg. = 2.8 bl/stm
# of Blocks Used During Patterning	783 bl/18 st Avg. = 43.5 bl/st	1019 bl/18 st Avg. = 56.6 bl/st	609 bl/17 st Avg. = 35.8 bl/st
Avg. # of Blocks Used to Make Each Pattern	783 bl/48 pat Avg. = 16.3 bl/pat	1019 bl/63 pat Avg. = 16.2 bl/pat	609 bl/46 pat Avg. = 13.3 bl/pat

Note. *N* = 18; one student was absent on Day 3.
Note. pat = patterns; st = student(s); bl = blocks/elements

To judge the complexity of the children's patterns, the team examined the number of elements the children used in their pattern stems. For example, an AB pattern has two elements, an AABB pattern has four elements, and an ABCD pattern also has four elements. (Fig. 2.2 shows a pattern with four elements in the pattern stem.) We found that three children used only two or three elements in their pattern stems every day, whereas all other children used four, five, six, eight, or even ten elements in their pattern stems. (Fig. 2.3 shows a pattern with ten elements in the pattern stem.) This analysis also showed that children used more elements in their pattern stems using the virtual pattern blocks than they did when drawing patterns or using the wooden pattern blocks. (See table 2.1.)

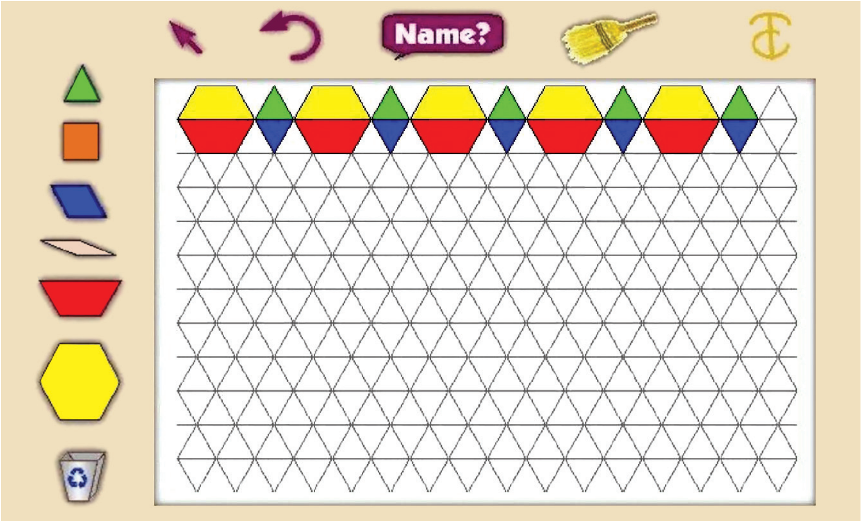

Fig. 2.2

Another examination looked at the total number of blocks the children used in each pattern. This analysis highlighted the work of children who may have completed a small number of patterns but who repeated those patterns numerous times. (Fig. 2.4 shows a child's pattern with numerous repeats.) This number was consistent with the number of patterns the children created. For example, there were more blocks incorporated on the day the children used the virtual pattern blocks than on the days they drew patterns or used wooden pattern blocks. However, when we examined the number of patterns created in relationship to the number of elements used to create those patterns, the number of elements in each pattern was consistent between virtual pattern blocks and wooden pattern blocks and slightly less for the drawings of blocks. (See table 2.1.) Children

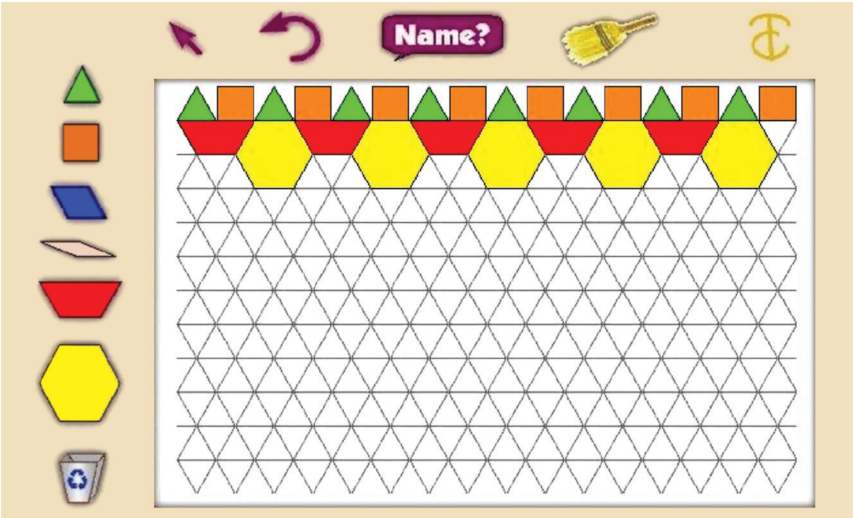

Fig. 2.3

seemed to lose interest in the pattern they were creating when they reached a certain number of blocks (approximately thirteen to sixteen) or repeats.

Another analysis examined patterns that went beyond simple repeating, left-to-right, horizontal patterns. The team classified several patterning

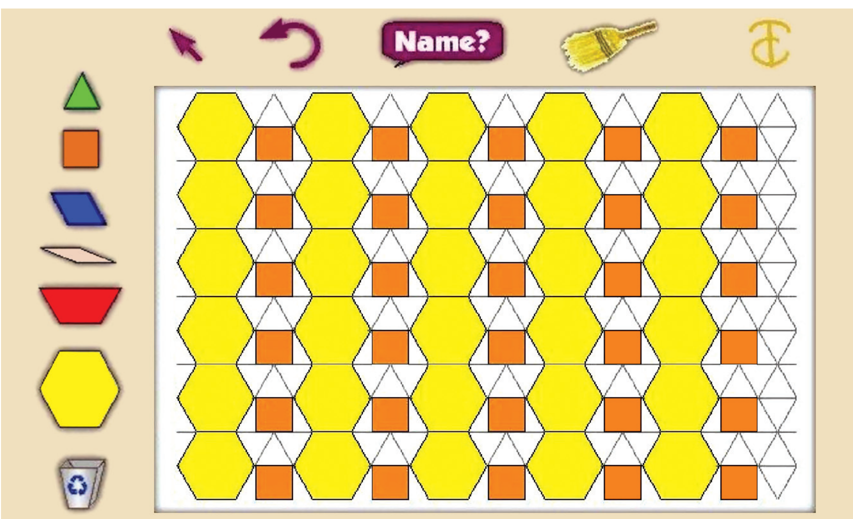

Fig. 2.4

behaviors as exhibiting “creativity.” These creative pattern behaviors are summarized in table 2.2. Some of these creative pattern behaviors included standing the wooden pattern blocks on end or stacking them to create tessellating patterns. (Being able to stack and build with the blocks is one of the advantages of using the wooden blocks in making patterns.) More than half of the children used the virtual pattern blocks to create new shapes by using one block to partially cover another. (Fig. 2.5 shows how a child used a hexagon to cover a rhombus and create a blue triangle.) Other creative behaviors included making vertical patterns or tessellations with the virtual pattern blocks. (Fig. 2.6 shows a child’s tessellation.) Children also made growing patterns with each different form of representation. (Fig. 2.7 shows a child’s growing pattern with the virtual pattern blocks.)

The final analysis examined overall similarities and differences among the three representational forms. We found that children created approximately the same number of patterns when drawing or using the wooden pattern blocks, but they created a greater number of patterns with the virtual pattern blocks. They used more elements in the pattern stem and exhibited more behaviors that the team classified as creative when using the virtual pattern blocks. Overall, the team identified several advantages for the children when they had the opportunity to create the patterns using the virtual pattern blocks. However, each form of representation allowed these kindergarten learners many opportunities to communicate the complexity of their thinking.

Table 2.2
Creative Pattern Behaviors—Kindergarten

Pattern Behaviors Classified as “Creative”	Day 1 Wooden Pattern Blocks	Day 2 Virtual Pattern Blocks	Day 3 Children’s Drawings
Standing Blocks on End	6 st		
Creating a New Shape		11 st	
Vertical Pattern		2 st	
Tessellating Pattern	6 st	9 st	
Growing Pattern	2 st	2 st	4 st
Totals	14	24	4

Note. N = 18; one student was absent on Day 3.
Note. st = student(s)

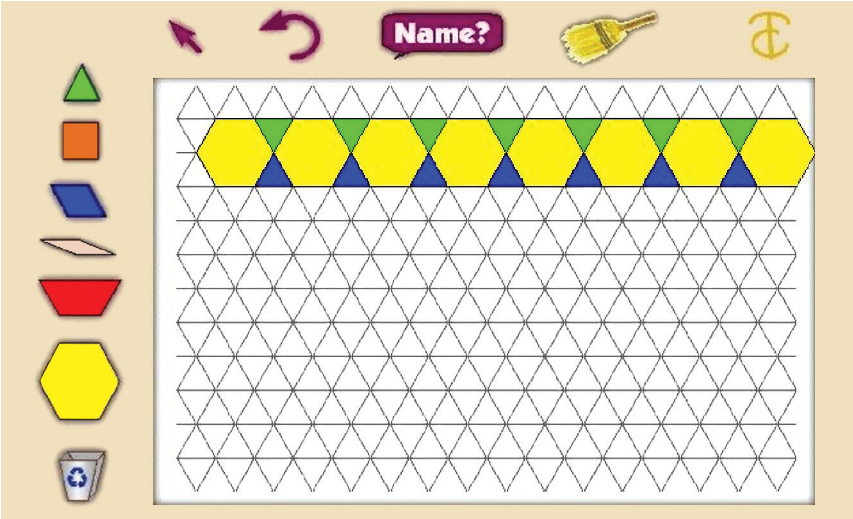

Fig. 2.5

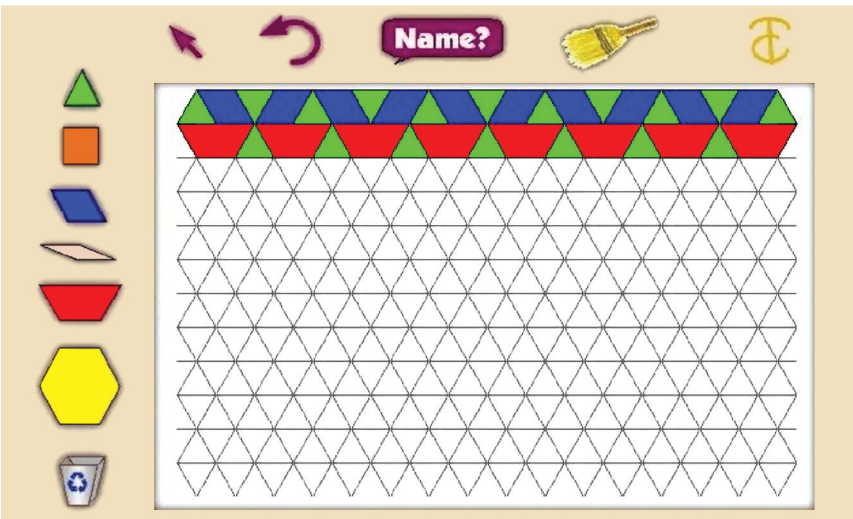

Fig. 2.6

Second-Graders Regrouping with Virtual Base-Ten Blocks

The second-grade class was composed of nineteen children, fifteen of whom spoke a language other than English in their homes. The group includ-

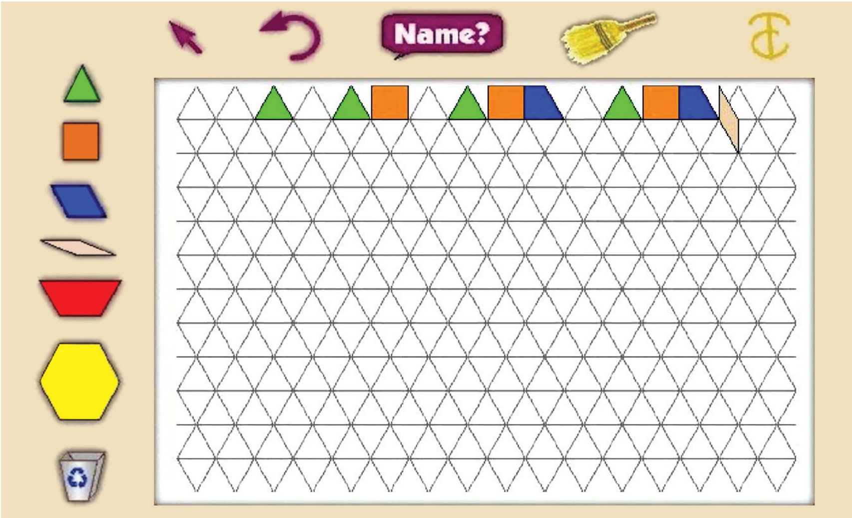

Fig. 2.7

ed Hispanic, Asian, and Caucasian students. During the two-day lesson on addition, the children focused on the regrouping of one- and two-digit numbers using virtual base-ten blocks, drawings, and written explanations. The class had previous experiences using the wooden base-ten blocks for number operations, and the use of these blocks was very effective for learning the process of trading ones blocks for tens blocks to model regrouping. The teacher was particularly interested in finding out if manipulating the virtual base-ten blocks on Day 1 would enhance the children's abilities to develop their visual-representation skills and to make drawings of the regrouping process on Day 2. The ability to understand ways of representing numbers and meanings of operations is an important part of the Standards for second-grade mathematics (NCTM 2000).

On Day 1 of the lesson, children used virtual base-ten blocks. Virtual base-ten blocks are available on several Web sites, including NCTM (nctm.org), Arcytech (www.arcytech.org/), and the National Library of Virtual Manipulatives (www.matti.usu.edu/nlvm). The virtual base-ten blocks are an electronic replica of the concrete base-ten blocks with representations of hundreds, tens, and ones blocks on a place-value mat. Children can group ten ones to make a ten by gluing them together, and they can break apart a tens rod to make ten ones. As the children used the virtual base-ten blocks, they recorded the sums for several one- and two-digit addition exercises. (Fig. 2.8 shows a child's work representing 68 + 15.) On

Day 1, many of the children lined up the ones cubes in a straight line, counted them to confirm there were ten, glued ten ones to make a tens rod, and moved the newly created tens rod to the tens column. This process was observable and allowed all children, particularly those who were second-language learners, to demonstrate and explain the process they were using to the observers.

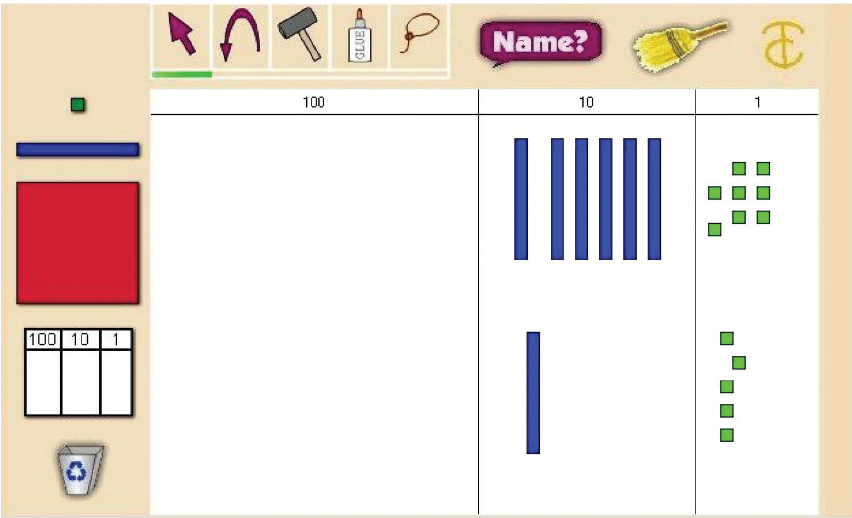

Fig. 2.8

On Day 2, the children received four pieces of paper, each with a different addition exercise. Each paper had one addition exercise, space for the children to draw a solution, and room to provide a written explanation. The teacher encouraged children to draw a representation of the addition processes on their papers for each addition exercise. The team was particularly interested in how the children would draw and explain their processes for regrouping ones to form a ten. To solve the problem 46 + 6, one child wrote: "I put 4 in the 10s place and 6 in the ones place and I put 6 more in the ones place. It was 12. I put 10 in a circle. I put the ten in the tens place." (Fig. 2.9 shows her drawing.)

As observers monitored the children's drawings during class, they noted that children circled ten ones on their papers and used arrows to simulate moving ten ones to the tens place as they had done the previous day with the virtual base-ten blocks. (Fig. 2.10 shows a child's drawing with arrows.) Some of the children also drew a new tens rod on their papers to show that ten ones became

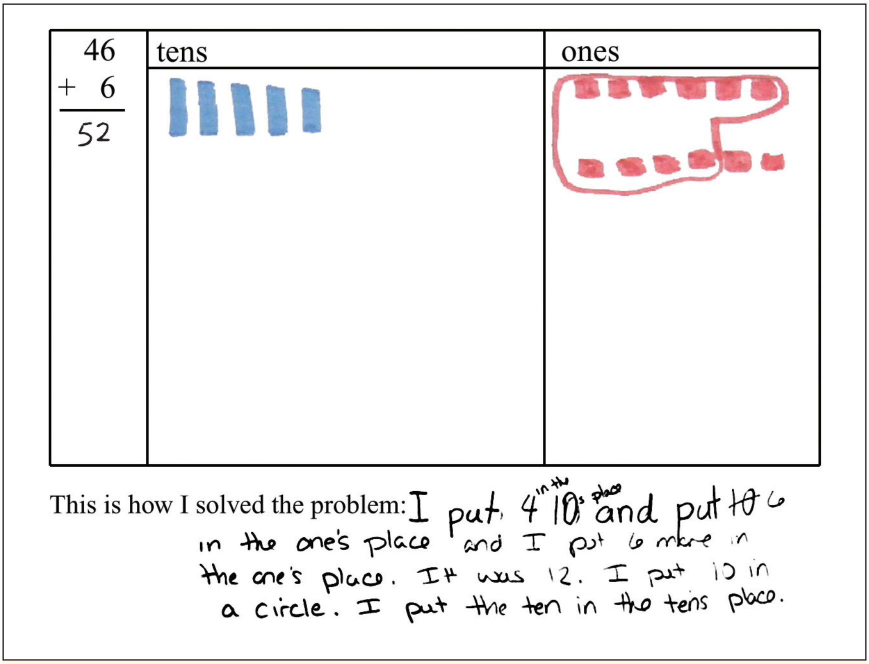

Fig. 2.9

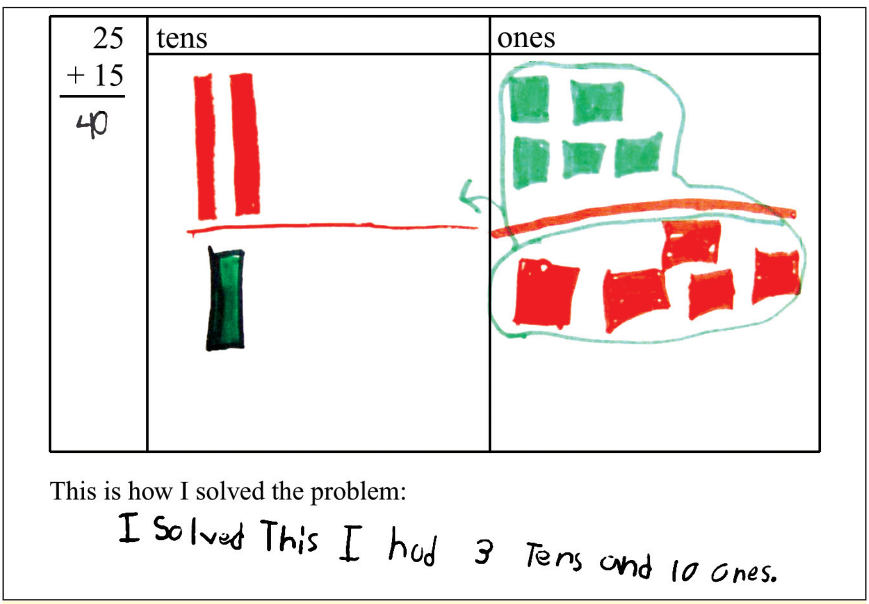

Fig. 2.10

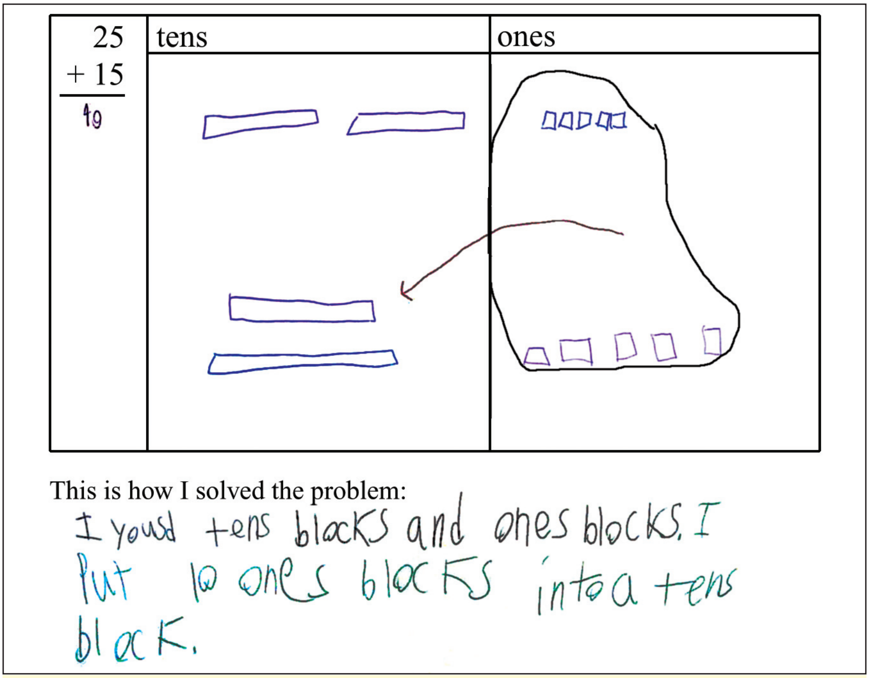

Fig. 2.11

one ten. To explain this, one child wrote, "I used tens blocks and ones blocks. I put 10 ones blocks into a tens block." (Fig. 2.11 shows his drawing of a new tens rod.) When solving the addition exercise 25 + 15, one child wrote, "I drew 3 tens then I put two fives at ones. I knew it was 40 because there were 30 then 5 + 5 = 10. So you turn the fives into a ten so it's 40. Goodbye." The children's drawings and explanations seemed to show that the virtual base-ten models offered a visual image the children could use to demonstrate the addition process on Day 2. Their explanations also indicated that they were developing flexibility with composing and decomposing numbers.

To analyze the children's work, the project team examined how the children explained and represented the concept and procedures for addition with regrouping. We calculated the percents of the children's correct answers on both days. These results are summarized in table 2.3. On Day 1, children worked in the computer lab and completed a worksheet with twelve addition exercises. The children completed 83 percent of the twelve addition exercises on the worksheet with 75 percent accuracy. On Day 2, children worked in the classroom and completed four addition exercises. For each exercise the children wrote the symbolic answer, drew a picture showing the addition

Table 2.3
Addition Problem Completion Rates and Strategy Use – Grade 2

	Day 1 Virtual Base-10 Blocks	Day 2 Drawings & Explanations
Problem Completion		
Number of Exercises Given	12	4
Percent of Exercises Completed	83%	80%
Accuracy of Symbolic Computation	75%	75%
Accuracy in Drawings/Explanations	NA	88%
Strategies Used by Students		
Counting	2	0
Counting & Place Value	6	0
Place Value	7	13
Place Value & Algorithm	4	6

Note. $N = 19$

process, and wrote an explanation for their drawings and symbols. The children completed 80 percent of the four addition exercises with 75 percent accuracy. However, they completed the drawings and written explanations with 88 percent accuracy. The reason for the discrepancy between the accuracy of the exercises only and the accuracy of the drawings and explanations was because many of the children drew an accurate picture of the addition exercise and wrote an accurate explanation for the problem; however, they did not complete the entire task by writing the sum for the addition exercise. Others drew an accurate representation with a correct explanation and wrote the wrong sum.

The team was also interested in whether the children used a counting method, a place-value-based regrouping method, or some other strategy. To document this process, the observers talked with children and examined their writings and drawings on both days. This part of the analysis indicated that on Day 1 (using virtual base-ten blocks), two children were simply counting; six children used both counting and a place-value strategy; seven children used only a place-value strategy; and four children used a place-value strategy with the traditional algorithm. On Day 2, thirteen children used a place-value strategy, and six children used a place-value strategy with the traditional algorithm. (See table 2.3.) For example, the child's work and written explanation in figure 2.12 shows that she understands the difference between numbers in the tens and ones places. Her explanation states, "I [took] 2 ten(s) to make 20, then I put 3 on the one(s). Then I look. I put 8 on the one(s). Then

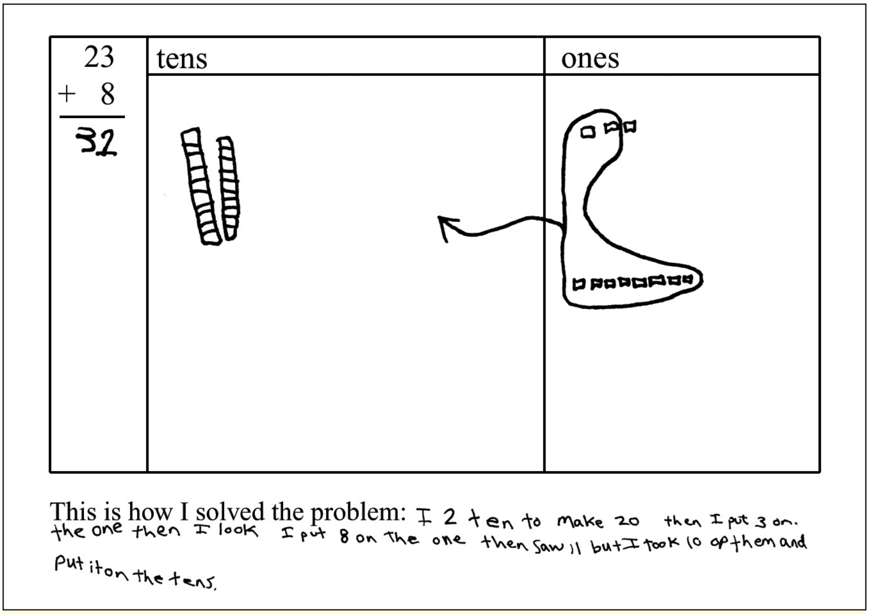

Fig. 2.12

I saw 11, but I took 10 of them and put it on the tens." We believe that the visual model (the virtual base-ten blocks) helped the children to "see" the regrouping of the numbers during the addition process, which gave this process more meaning for the children.

The team noticed unique methods children used to solve the exercises. On Day 1 (virtual base-ten blocks), three of the children used eleven ones for the numeral 11 instead of using a single tens rod and a single ones unit. Another unique method used by three of the higher-ability children was a more efficient grouping strategy. Using the Web site's grouping tool, called a lasso, the children generated groups of ones, lassoed the ones, and then moved them to the ones column as a group instead of moving one block at a time as most of the other children had done. (Fig. 2.13 shows a child using the lasso tool on the Web site.) Some children used the tens rod as a reference by lining up ones units next to it to confirm that they had ten ones. (Fig. 2.14 shows how a child lines up ones against the tens rod.)

On Day 2 (drawings and explanations), children represented the regrouping process in unique ways. For example, some children used two different colors to represent ones and tens. One child even used two colors for two different sets of ones to show how the ones joined to make one ten. (Fig. 2.15

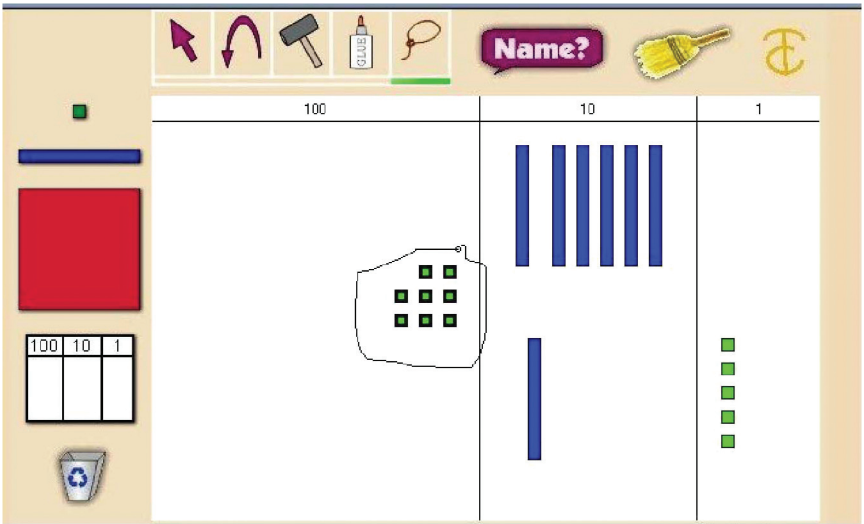

Fig. 2.13

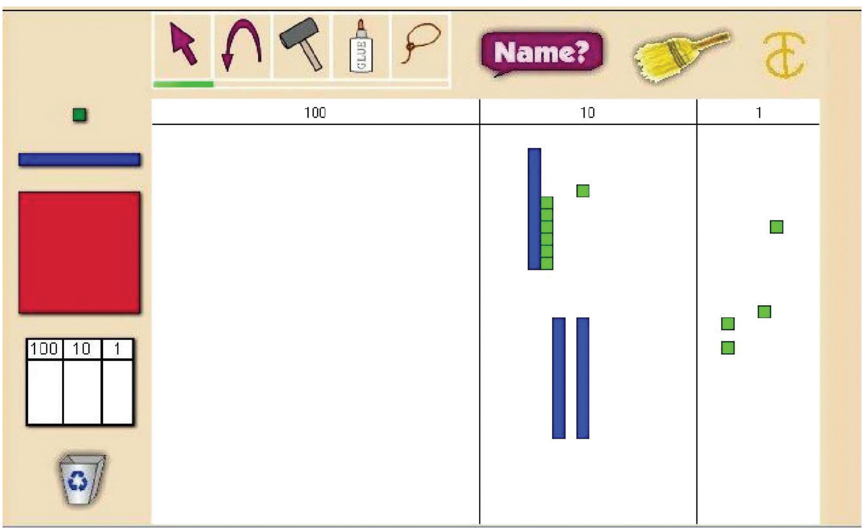

Fig. 2.14

shows this representation for 39 + 23.) The tens and ones for the number 39 are drawn in pink and the tens and ones for the number 23 are drawn in blue. To show the 9 ones and the 1 one joining together to make a ten, the child drew the newly created ten in both pink and blue. We believe that these unique

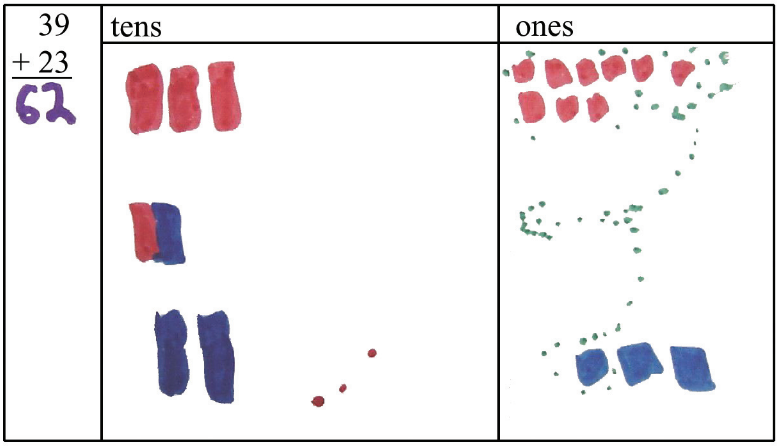

Fig. 2.15

drawings and explanations showed the extent of children's understanding of the addition process.

Research on children's understanding of place value indicates that many children think that the 1 in 15 means one until third or fourth grade (Kamii 1980; Ross 1986). During this project, the visual images of tens and ones seemed to help the children develop meaning for numbers in the ones and tens places. During the weeks following this two-day lesson, the second-grade teacher noticed the relative ease with which the children transferred their understanding to nonpictorial addition exercises.

Closing Comments

The teachers in these projects used the virtual manipulatives to make important connections among (1) the concrete manipulations of objects, (2) the visual images of those objects, (3) abstract mathematical ideas and notations, and (4) the processes underlying these concepts. The virtual manipulatives provided the children with flexible representations that allowed them to explore their ideas. The virtual environment enabled the children to test their mathematical ideas as they were working with the virtual tools. Young children, particularly second-language learners, may have difficulty verbalizing their mathematical thinking. In these classroom projects, using virtual manipulatives helped children clarify their own thinking through experimentation and allowed them to communicate and demonstrate their mathematical thinking to others.

In these classrooms, teachers used the concrete manipulatives (pattern blocks and base-ten blocks) prior to using the virtual manipulatives. The use of the virtual blocks during these projects provided an important bridge for the children between the use of concrete objects and the use of pictorial and symbolic notations for representing abstract mathematical ideas. Some virtual manipulatives offer concept tutorials that link objects and manipulations on those objects with the abstract symbols that represent them. Making this link among representational forms explicit is an important feature of using virtual manipulatives with children. Although these projects examined only two classrooms of children, they document important preliminary findings about the usefulness of virtual manipulatives for teaching mathematics with young children. We believe it is important to investigate problems and questions using different technologies and forms of representations in real classrooms and to explore effective ways to use these technologies in teaching mathematics to all children.

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