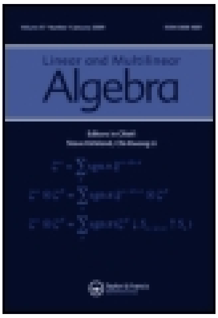

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New regions including eigenvalues of Toeplitz matrices

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New regions including eigenvalues of Toeplitz matrices

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Two new eigenvalue inclusion regions for matrices with a constant main diagonal are given. We then apply these results to Toeplitz matrices, and obtain two regions including all eigenvalues of Toeplitz matrices. Furthermore, it is proved that the new regions are tighter than those in [Melman A. Ovals of Cassini for Toeplitz matrices, *Linear and Multilinear Algebra*. 2012;60:189–199].

Keywords: eigenvalue; Toeplitz; Geršgorin; Brauer

AMS Subject Classifications: 15A18; 15B05

1. Introduction

Eigenvalue localization has been a hot topic in matrix theory and its applications. Many researchers have obtained lots of eigenvalue inclusion regions; for details, see [1–3]. One of the most famous results is given by Geršgorin [6] as follows:

THEOREM 1.1 [6] *Let $A = [a_{ij}] \in C^{n\times n}$ and $\sigma(A)$ be the spectrum of A. Then*

$$\sigma(A) \subseteq \Gamma(A) = \bigcup_{i\in N} \Gamma_i(A),$$

where $N = \{1, 2, \dots, n\}$ and $\Gamma_i(A) = \{z \in C : \ |z - a_{ii}| \le r_i(A) = \sum_{k\neq i} |a_{ik}|\}$.

We here call $\Gamma(A)$ the Geršgorin set of A. To capture all eigenvalues of matrices more precisely than the Geršgorin set, Brauer [1] provided another well-known eigenvalue inclusion region.

THEOREM 1.2 [1] *Let $A = [a_{ij}] \in C^{n\times n}$, $n \ge 2$. Then*

$$\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{\substack{i,j\in N,\\ i\neq j}} \mathcal{K}_{i,j}(A),$$

where $\mathcal{K}_{i,j}(A) = \{z \in C : |z - a_{ii}||z - a_{jj}| \le r_i(A) r_j(A)\}$.

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The set $\mathcal{K}(A)$ is called the Brauer set of A. It is well known that $\mathcal{K}(A) \subseteq \Gamma(A)$; see [12,13]. Since A and its transpose A^T have the same spectrum, we have that $\sigma(A) = \sigma(A^T) \subseteq \mathcal{K}(A^T) \subseteq \Gamma(A^T)$.

Recently, Melman gave some new eigenvalue inclusion regions for some special classes of matrices by taking into account their structure.[8–11] In particular, for a matrix $A = [a_{ij}]$ with a constant main diagonal entry $\bar{a}$, i.e. $a_{ii} = \bar{a}$ for all $i \in N$, Melman pointed out that the Geršgorin set $\Gamma(A)$ and the Brauer set $\mathcal{K}(A)$ each consist of a single disc, that is,

$$\Gamma(A) = \left\{ z \in C : \ |z - \bar{a}| \leq \max_{i \in N} r_i(A) \right\},$$

$$\mathcal{K}(A) = \left\{ z \in C : |z - \bar{a}| \leq \max_{\substack{i,j \in N, \\ i \neq j}} \sqrt{r_i(A) r_j(A)} \right\},$$

and gave a new region (see Theorem 1.3), which is contained in $\Gamma(A)$ and $\mathcal{K}(A)$.

THEOREM 1.3 [9] *Let $A = [a_{ij}] \in C^{n \times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \Omega(A) = \bigcup_{i \in N} \Omega_i(A),$$

where $A_0 = A - \bar{a}I$, $(A_0^2)_{ij}$ denotes the (i, j)-th entry of A_0^2 and

$$\Omega_i(A) = \left\{ z \in C : \left| z - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}} \right| \left| z - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}} \right| \leq r_i\left(A_0^2\right) \right\}.$$

Furthermore, $\Omega(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A)$.

Toeplitz matrices, as one class of structured matrices with a constant main diagonal which is frequently encountered in numerical analysis and in digital signal and image processing, are constant along all their NW-SE diagonals,[9] that is, a Toeplitz matrix $T \in C^{n \times n}$ has the following form:

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{2-n} & \cdots & t_{-1} & t_0 & t_1 \\ t_{1-n} & \cdots & t_{-2} & t_{-1} & t_0 \end{bmatrix}.$$

Also in [9], Melman applied Theorem 1.3 to Toeplitz matrices, and got the following simple form.

THEOREM 1.4 [9] *Let $T \in C^{n \times n}$ be a Toeplitz matrix with $t_{11} = \bar{t}$, $n \geq 2$. Then*

$$\sigma(T) \subseteq \Omega(T) = \bigcup_{i=1}^{\lceil \frac{n}{2} \rceil} \left\{ z \in C : \left| z - \bar{t} - \sqrt{\left(T_0^2\right)_{ii}} \right| \left| z - \bar{t} + \sqrt{\left(T_0^2\right)_{ii}} \right| \leq v_i\left(T_0^2\right) \right\},$$

where $T_0 = T - \bar{t}I$, $v_i\left(T_0^2\right) = \max\left\{ r_i\left(T_0^2\right), \ r_{n-i+1}\left(T_0^2\right) \right\}$ and

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2}, & if\ n\ is\ even, \\ \frac{n+1}{2}, & if\ n\ is\ odd. \end{cases}$$

Furthermore, $\Omega(T) \subseteq \mathcal{K}(T) \subseteq \Gamma(T)$.

In this paper, we continue to research the problem of eigenvalue localization for matrices with a constant main diagonal. In Section 2, we give two new regions including eigenvalues by establishing sufficient conditions for non-singular matrices, and then apply the results to Toeplitz matrices, and obtain two eigenvalue inclusion regions for Toeplitz matrices in Section 3. All new eigenvalue inclusion regions are proved to be tighter than those in [9].

2. Eigenvalue inclusion regions for matrices with a constant main diagonal

In this section, we give two new regions including all eigenvalues of matrices with a constant main diagonal. Before that, two sufficient conditions for non-singular matrices are given.

LEMMA 2.1 [12, Theorem 1.16] *For any* $A = [a_{ij}] \in C^{n\times n}$, *if there exists* $\alpha \in [0, 1]$ *such that for all* $i \in N$,

$$|a_{ii}| > (r_i(A))^{\alpha}(c_i(A))^{1-\alpha},$$

where $c_i(A) = r_i(A^T)$, *then A is non-singular.*

THEOREM 2.2 *For any* $A = [a_{ij}] \in C^{n\times n}$ *with* $a_{ii} = \bar{a}$ *for all* $i \in N$, *if there exists* $\alpha \in [0, 1]$ *such that for all* $i \in N$,

$$\left|\bar{a}^2 - \left(A_0^2\right)_{ii}\right| > \left(r_i\left(A_0^2\right)\right)^{\alpha}\left(c_i\left(A_0^2\right)\right)^{1-\alpha}, \tag{1}$$

where $A_0 = A - \bar{a}I$, *then A is non-singular.*

Proof Suppose on the contrary that A is singular. Then there is a vector x with $x \neq 0$ such that $Ax = 0$. Note that $A_0 = A - \bar{a}I$, we have

$$A_0x = -\bar{a}x,$$

which leads to

$$A_0^2x = \bar{a}^2x,$$

equivalently, $\left(A_0^2 - \bar{a}^2I\right)x = 0$, which implies that $A_0^2 - \bar{a}^2I$ is singular. Let $B = [b_{ij}] = A_0^2 - \bar{a}^2I$, where $b_{ii} = \left(A_0^2\right)_{ii} - \bar{a}^2$ for $i \in N$, and $b_{ij} = \left(A_0^2\right)_{ij}$ for $i \neq j$. Then B is singular. Furthermore, from Equation (1), we have

$$|b_{ii}| > (r_i(B))^{\alpha}(c_i(B))^{1-\alpha},$$

which, by Lemma 2.1, implies that B is non-singular. This is a contradiction. Thus A is v. □

From Theorem 2.2 and the generalized arithmetic–geometric mean inequality:

$$\alpha a + (1-\alpha)b \geq a^{\alpha}b^{1-\alpha}$$

where $a, b \geq 0$ and $0 \leq \alpha \leq 1$, we easily get another sufficient condition for non-singular matrices.

THEOREM 2.3 *For any $A = [a_{ij}] \in C^{n\times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, if there exists $\alpha \in [0, 1]$ such that for all $i \in N$,*

$$\left|\bar{a}^2 - \left(A_0^2\right)_{ii}\right| > \alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right),$$

where $A_0 = A - \bar{a}I$, then A is non-singular.

As is well known, a sufficient condition for non-singular matrices leads to an eigenvalue inclusion region, and vice versa.[3] From Theorems 2.2 and 2.3, we can obtain new eigenvalue inclusion regions for matrices with a constant main diagonal.

THEOREM 2.4 *Let $A = [a_{ij}] \in C^{n\times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \Omega^1(A) = \bigcap_{0\leq\alpha\leq 1}\bigcup_{i\in N}\Omega_i^{1\alpha}(A),$$

where

$$\Omega_i^{1\alpha}(A) = \left\{z \in C : \left|z - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right|\left|z - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| \leq \alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right)\right\}.$$

Proof Suppose that $\lambda \in \sigma(A)$, then $\lambda I - A$ is singular. If $\lambda \notin \Omega^1(A)$, then there exists $\alpha \in [0, 1]$ such that for any $i \in N$, $\lambda \notin \Omega_i^{1\alpha}(A)$, which implies that for any $i \in N$,

$$\left|\lambda - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right|\left|\lambda - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| > \alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right),$$

that is,

$$\left|(\lambda - \bar{a})^2 - \left(\sqrt{\left(A_0^2\right)_{ii}}\right)^2\right| = \left|(\lambda - \bar{a})^2 - \left(A_0^2\right)_{ii}\right| > \alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right). \quad (2)$$

From Equation (2) and Theorem 2.3, we have that $\lambda I - A$ is non-singular. This is a contradiction to that $\lambda I - A$ is singular. Hence, $\lambda \in \Omega^1(A)$. □

Similar to the proof of Theorem 2.4, we can easily prove the following Theorem by Theorem 2.2.

THEOREM 2.5 *Let $A = [a_{ij}] \in C^{n\times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \Omega^2(A) = \bigcap_{0\leq\alpha\leq 1}\bigcup_{i\in N}\Omega_i^{2\alpha}(A),$$

where

$$\Omega_i^{2\alpha}(A) = \left\{z \in C : \left|z - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right|\left|z - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| \leq \left(r_i\left(A_0^2\right)\right)^{\alpha}\left(c_i\left(A_0^2\right)\right)^{1-\alpha}\right\}.$$

□

We now establish Next, the comparison between $\Omega^1(A)$ with $\Omega^2(A)$.

THEOREM 2.6 *Let $A = [a_{ij}] \in C^{n\times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, $n \geq 2$. Then*

$$\Omega^2(A) \subseteq \Omega^1(A).$$

Proof Let $z_0 \notin \Omega^1(A)$. Then there exists $\alpha \in [0, 1]$ such that for any $i \in N$, $z_0 \notin \Omega_i^{1\alpha}(A)$, that is,

$$\left|z_0 - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right| \left|z_0 - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| > \alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right).$$

Furthermore, note that

$$\alpha r_i\left(A_0^2\right) + (1-\alpha)c_i\left(A_0^2\right) \geq \left(r_i\left(A_0^2\right)\right)^{\alpha}\left(c_i\left(A_0^2\right)\right)^{1-\alpha}.$$

Hence, for any $i \in N$,

$$\left|z_0 - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right| \left|z_0 - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| > \left(r_i\left(A_0^2\right)\right)^{\alpha}\left(c_i\left(A_0^2\right)\right)^{1-\alpha},$$

which implies that $z_0 \notin \Omega^2(A)$. The conclusion follows by the fact that if $z_0 \notin \Omega^1(A)$, then $z_0 \notin \Omega^2(A)$. □

Next, the comparisons between $\Omega^1(A)$, $\Omega^2(A)$, $\Omega(A)$, $\Gamma(A)$ and $\mathcal{K}(A)$ are given.

THEOREM 2.7 *Let $A = [a_{ij}] \in C^{n\times n}$ with $a_{ii} = \bar{a}$ for all $i \in N$, $n \geq 2$. Then*

$$\Omega^2(A) \subseteq \Omega^1(A) \subseteq (\Omega(A)\bigcap\Omega(A^T)) \subseteq (\mathcal{K}(A)\bigcap\mathcal{K}(A^T)) \subseteq (\Gamma(A)\bigcap\Gamma(A^T)).$$

Proof From Theorems 1.3 and 2.6, we need only to prove that $\Omega^1(A) \subseteq (\Omega(A)\bigcap\Omega(A^T))$. In fact, when $\alpha = 1$, $\Omega_i^{1\alpha}(A) = \Omega_i(A)$, consequently, $\Omega^1(A) = \Omega(A)$. And when $\alpha = 0$, $\Omega_i^{1\alpha}(A) = \Omega_i(A^T)$, consequently, $\Omega^1(A) = \Omega(A^T)$. Hence, $\Omega^1(A) \subseteq (\Omega(A)\bigcap \Omega(A^T))$. □

Remark 1 The sets $\Omega^1(A)$ and $\Omega^2(A)$ are of interest theoretically because it provides a tighter set containing all the eigenvalues of a matrix with a constant main diagonal. However, they are not of much practical use because of the restriction of α. To give more convenient forms of $\Omega^1(A)$ and $\Omega^2(A)$, we use the method provided in [4,7], and obtain easily the following equivalent forms of $\Omega^1(A)$ and $\Omega^2(A)$, namely,

$$\Omega^1(A) = \overline{\Omega}(A)\bigcup\widehat{\Omega}(A), \tag{3}$$

where $\overline{\Omega}(A) = \bigcup_{i\in N}\overline{\Omega}_i(A)$, and

$$\overline{\Omega}_i(A) = \left\{z \in C : \left|z - \bar{a} - \sqrt{\left(A_0^2\right)_{ii}}\right| \left|z - \bar{a} + \sqrt{\left(A_0^2\right)_{ii}}\right| \leq \min\left\{r_i\left(A_0^2\right), c_i\left(A_0^2\right)\right\}\right\},$$

$$\widehat{\Omega}(A) = \bigcup_{\substack{i\in\mathcal{R},\\ j\in\mathcal{C}}}\widehat{\Omega}_{ij}(A), \text{ and}$$

$$\widehat{\Omega}_{ij}(A) = \Bigg\{ z \in C : \left|z - \bar{a} - \sqrt{(A_0^2)_{ii}}\right| \left|z - \bar{a} + \sqrt{(A_0^2)_{ii}}\right| \left(c_j\left(A_0^2\right) - r_j\left(A_0^2\right)\right) + \left|z - \bar{a} - \sqrt{(A_0^2)_{jj}}\right| \left|z - \bar{a} + \sqrt{(A_0^2)_{jj}}\right| \left(r_i\left(A_0^2\right) - c_i\left(A_0^2\right)\right) \le c_j\left(A_0^2\right) r_i\left(A_0^2\right) - c_i\left(A_0^2\right) r_j\left(A_0^2\right), i \in \mathcal{R},\ j \in \mathcal{C} \Bigg\};$$

$$\Omega^2(A) = \overline{\Omega}(A) \bigcup \widetilde{\Omega}(A), \tag{4}$$

where $\widetilde{\Omega}(A) = \bigcup_{\substack{i \in \mathcal{R}, \\ j \in \mathcal{C}}} \widetilde{\Omega}_{ij}(A)$, and

$$\widetilde{\Omega}_{ij}(A) = \Bigg\{ z \in C : \frac{\left|z - \bar{a} - \sqrt{(A_0^2)_{ii}}\right| \left|z - \bar{a} + \sqrt{(A_0^2)_{ii}}\right|}{c_i\left(A_0^2\right)} \bullet \left(\frac{\left|z - \bar{a} - \sqrt{(A_0^2)_{jj}}\right| \left|z - \bar{a} + \sqrt{(A_0^2)_{jj}}\right|}{c_j\left(A_0^2\right)} \right)^{\log_{\frac{c_j(A_0^2)}{r_j(A_0^2)}} \frac{r_i(A_0^2)}{c_i(A_0^2)}} \le 1,\ i \in \mathcal{R}\backslash\{l : c_l(A_0^2) = 0\},\ j \in \mathcal{C}\backslash\{k : r_k(A_0^2) = 0\} \Bigg\}.$$

Here,

$$\mathcal{R} = \left\{ i \in N : r_i(A_0^2) > c_i(A_0^2) \right\}, \text{ and } \mathcal{C} = \left\{ i \in N : c_i(A_0^2) > r_i(A_0^2) \right\}.$$

Example 2.8 Consider the following matrix A (A_4 in [9]),

$$A = \begin{bmatrix} 2 & i & -3 & -i \\ 0 & 2 & 1 & -5i \\ 4 & 1 & 2 & 2 \\ i & -1 & 1 & 2 \end{bmatrix}.$$

The Geršgorin set $\Gamma(A)$ ($\Gamma(A^T)$), the Brauer set $\mathcal{K}(A)$ ($\mathcal{K}(A^T)$) and the set $\Omega(A)$, ($\Omega(A^T)$) are shown in Figure 1 (Figure 2, respectively), where $\Gamma(A)$ ($\Gamma(A^T)$) is represented by the outside boundary, $\mathcal{K}(T)$ ($\mathcal{K}(A^T)$) by the inner and $\Omega(T)$ ($\Omega(A^T)$) is filled. $\Omega^1(A)$ and $\Omega^2(A)$ are shown in Figures 3 and 4, respectively. The exact eigenvalues are plotted with asterisks. It is easy to see that

$$\Omega^2(A) \subset \Omega^1(A) \subset (\Omega(A) \bigcap \Omega(A^T)) \subset (\mathcal{K}(A) \bigcap \mathcal{K}(A^T)) \subset (\Gamma(A) \bigcap \Gamma(A^T)).$$

This example shows that the two new eigenvalue inclusion regions in Theorems 2.4 and 2.5 are smaller than the intersection of the sets $\Omega(A)$ and $\Omega(A^T)$ obtained in [9], consequently, smaller than the intersection of the well-known Geršgorin and Brauer sets of a matrix with a constant main diagonal entry and its transpose.

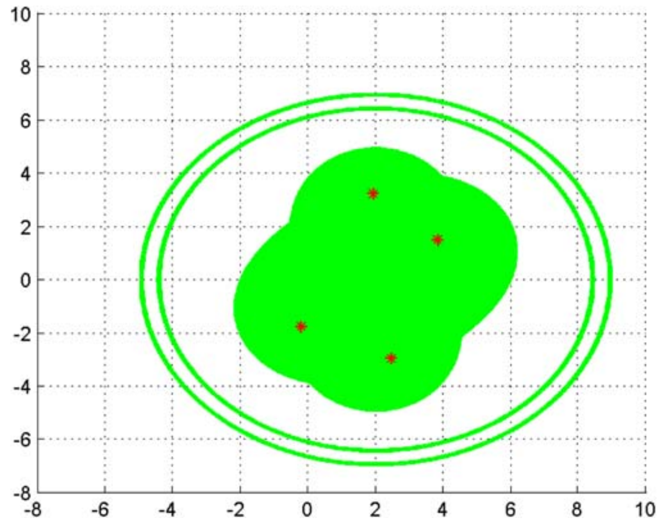

Figure 1. $\Omega(A) \subset \mathcal{K}(A) \subset \Gamma(A)$.

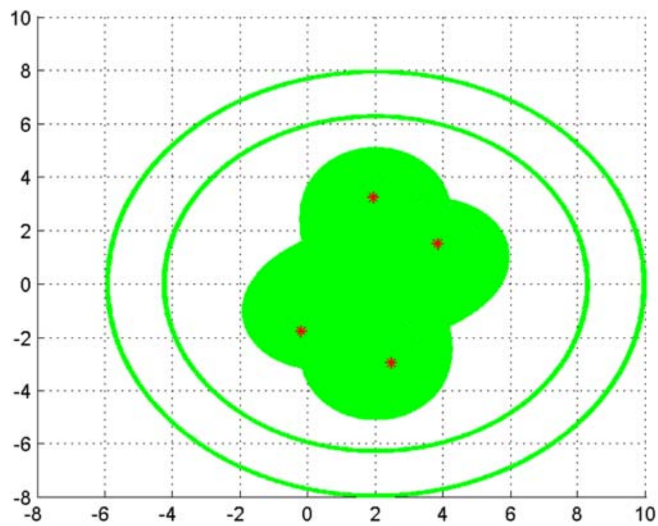

Figure 2. $\Omega(A^T) \subset \mathcal{K}(A^T) \subset \Gamma(A^T)$.

3. Eigenvalue inclusion regions for Toeplitz matrices

In this section, we apply Theorems 2.4 and 2.5 to Toeplitz matrices, and obtain a much simpler form of the eigenvalue inclusion theorem for Toeplitz matrices by considering its structure. Firstly, we recall some results on Toeplitz matrices introduced in [9]. A Toeplitz matrix is persymmetric. Here, we call A persymmetric if A is symmetric with respect to the main anti-diagonal.[9] The square of a Toeplitz matrix T is not necessary Toeplitz, but

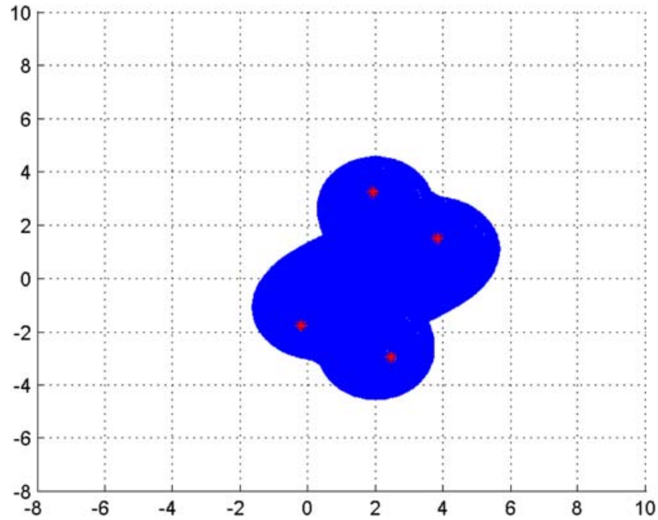

Figure 3. $\Omega^1(A)$.

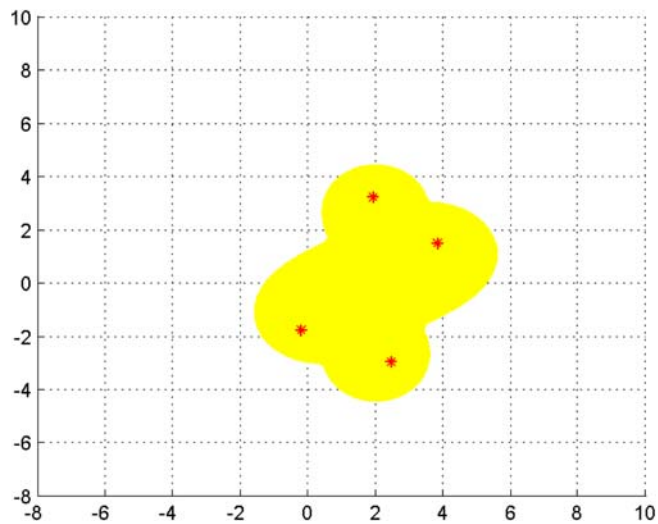

Figure 4. $\Omega^2(A)$.

it is persymmetric. Hence, the main diagonal of T^2 has at most $\lceil \frac{n}{2} \rceil$ distinct values, and $r_i(T^2) = c_{n-i+1}(T^2)$. Obviously, when n is odd, then $\lceil \frac{n}{2} \rceil = n - \lceil \frac{n}{2} \rceil + 1$, consequently, $r_{\lceil \frac{n}{2} \rceil}(T^2) = c_{n-\lceil \frac{n}{2} \rceil+1}(T^2)$.

THEOREM 3.1 *Let $T = [t_{ij}] \in C^{n\times n}$ be a Toeplitz matrix with $t_{11} = \bar{t}$, $n \geq 2$. Then*

$$\sigma(T) \subseteq \Omega^1(T) = \left(\bigcup_{i=1}^{\lceil \frac{n}{2} \rceil} \overline{\Omega}_i(T)\right) \bigcup \left(\bigcup_{\substack{i\in\mathcal{R}, j\in\mathcal{C},\\ j\neq n-i+1}} \widehat{\Omega}_{ij}(T)\right) \bigcup \left(\bigcup_{i\in\mathcal{R}} \widehat{\Omega}_{i^1}(T)\right),$$

where $T_0 = T - \bar{t}I$, $\overline{\Omega}_i(T)$, $\widehat{\Omega}_{ij}(T)$ are defined as (3), and

$$\widehat{\Omega}_{i^1}(T) = \left\{ z \in C : \left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{ii}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{ii}}\right| \leq \frac{r_i(T_0^2) + c_i(T_0^2)}{2} \right\}.$$

Proof Since T is Toeplitz and $T_0 = T - \bar{t}I$, we have that T_0 is also Toeplitz, and T_0^2 is persymmetric. Thus,

$$r_i(T_0^2) = c_{n-i+1}(T_0^2), \tag{5}$$

and

$$(T_0^2)_{ii} = (T_0^2)_{n-i+1,n-i+1}. \tag{6}$$

By Theorem 2.4 and (3) , we get that for any $\lambda \in \sigma(T)$, $\lambda \in \Omega^1(T) = \overline{\Omega}(T) \bigcup \widehat{\Omega}(T)$. For $\overline{\Omega}(T)$, we easily obtain from Equations (5) and (6) that

$$\overline{\Omega}(T) = \bigcup_{i=1}^{\lceil \frac{n}{2} \rceil} \overline{\Omega}_i(T). \tag{7}$$

Moreover, if $i_0 \in \mathcal{R}$, then $n - i_0 + 1 \in \mathcal{C}$ from the fact that T_0^2 is persymmetric. Hence, the following inequality

$$\begin{aligned} &\left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \left(c_{n-i_0+1}\left(T_0^2\right) - r_{n-i_0+1}\left(T_0^2\right)\right) \\ &+ \left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{n-i_0+1,n-i_0+1}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{n-i_0+1,n-i_0+1}}\right| \left(r_{i_0}\left(T_0^2\right) - c_{i_0}\left(T_0^2\right)\right) \\ &\leq c_{n-i_0+1}\left(T_0^2\right) r_{i_0}\left(T_0^2\right) - c_{i_0}\left(T_0^2\right) r_{n-i_0+1}\left(T_0^2\right) \end{aligned}$$

is equivalent to

$$2\left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \left(r_{i_0}\left(T_0^2\right) - c_{i_0}\left(T_0^2\right)\right) \leq r_{i_0}^2\left(T_0^2\right) - c_{i_0}^2\left(T_0^2\right),$$

which, from $i_0 \in \mathcal{R}$, i.e. $r_{i_0}\left(T_0^2\right) > c_{i_0}\left(T_0^2\right)$, leads to

$$\left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{i_0 i_0}}\right| \leq \frac{r_{i_0}\left(T_0^2\right) + c_{i_0}\left(T_0^2\right)}{2}. \tag{8}$$

Combining the definition of $\widehat{\Omega}(T)$ with Inequality (8), we have

$$\widehat{\Omega}(T) = \left(\bigcup_{\substack{i\in\mathcal{R}, j\in\mathcal{C},\\ j\neq n-i+1}} \widehat{\Omega}_{ij}(T)\right) \bigcup \left(\bigcup_{i\in\mathcal{R}} \widehat{\Omega}_{i^1}(T)\right), \tag{9}$$

where

$$\widehat{\Omega}_{i^1}(T) = \left\{ z \in C : \left|z - \bar{t} - \sqrt{\left(T_0^2\right)_{ii}}\right| \left|z - \bar{t} + \sqrt{\left(T_0^2\right)_{ii}}\right| \leq \frac{r_i(T_0^2) + c_i(T_0^2)}{2} \right\}.$$

The conclusion follows by (3), (7) and (9). □

Similar to the proof of Theorem 3.1, we can obtain the following eigenvalue inclusion region for Toeplitz matrices, and omit its proof.

THEOREM 3.2 *Let $T = [t_{ij}] \in C^{n\times n}$ be a Toeplitz matrix with $t_{11} = \bar{t}$, $n \geq 2$. Then*

$$\sigma(T) \subseteq \Omega^2(T) = \left(\bigcup_{i=1}^{\lceil \frac{n}{2} \rceil} \overline{\Omega}_i(T)\right) \bigcup \left(\bigcup_{\substack{i\in\mathcal{R}, j\in\mathcal{C},\\ j\neq n-i+1}} \widetilde{\Omega}_{ij}(T)\right) \bigcup \left(\bigcup_{i\in\mathcal{R}} \widetilde{\Omega}_{i^2}(T)\right),$$

where $T_0 = T - \bar{t}I$, $\overline{\Omega}_i(T)$, $\widetilde{\Omega}_{ij}(T)$ are defined as (4), and

$$\widetilde{\Omega}_{i^2}(T) = \left\{ z \in C : \left| z - \bar{t} - \sqrt{\left(T_0^2\right)_{ii}} \right| \left| z - \bar{t} + \sqrt{\left(T_0^2\right)_{ii}} \right| \leq \sqrt{r_i(T_0^2)c_i(T_0^2)} \right\}.$$

From Theorems 1.4, 2.7, 3.1 and 3.2, and the fact that for a Toeplitz matrix T, $\Gamma(T) = \Gamma(T^T)$, $\mathcal{K}(T) = \mathcal{K}(T^T)$, $\Omega(T) = \Omega(T^T)$, we have the comparison results as follows.

COROLLARY 3.3 *Let $T = [t_{ij}] \in C^{n\times n}$ be a Toeplitz matrix with $t_{11} = \bar{t}$, $n \geq 2$. Then*

$$\Omega^2(T) \subseteq \Omega^1(T) \subseteq \Omega(T) \subseteq \mathcal{K}(T) \subseteq \Gamma(T).$$

Example 3.4 Consider the following Toeplitz matrix T (the matrix Q in [9])

$$T = \begin{bmatrix} 6 & 1 & -1 & -2i \\ 0 & 6 & 1 & -1 \\ -1 & 0 & 6 & 1 \\ 4 & -1 & 0 & 6 \end{bmatrix}.$$

In Figure 5, the sets $\Omega(T)$, $\mathcal{K}(T)$, and $\Gamma(T)$ are shown, where $\Gamma(T)$ is represented by the outside boundary, $\mathcal{K}(T)$ by the inner and $\Omega(T)$ is filled. The sets $\Omega^1(T)$ and $\Omega^2(T)$ are shown in Figures 6 and 7, respectively. The exact eigenvalues are plotted with asterisks. As we can see,

$$\Omega^2(T) \subset \Omega^1(T) \subset \Omega(T) \subset \mathcal{K}(T) \subset \Gamma(T).$$

This example shows that the two new eigenvalue inclusion regions in Theorems 3.1 and 3.2 are smaller than the set obtained in [9], the Geršgorin set and the Brauer set for a Toeplitz matrix.

Example 3.5 Consider the following Toeplitz matrices

$$T = \begin{bmatrix} 2 & i & i & 0 & \cdots & 0 & 0 & -i \\ -2i & 2 & i & i & \ddots & 0 & 0 & 0 \\ 0 & -2i & 2 & i & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 2 & i & i \\ 0 & 0 & 0 & 0 & \ddots & -2i & 2 & i \\ 1 & 0 & 0 & 0 & \cdots & 0 & -2i & 2 \end{bmatrix} \in C^{n\times n},$$

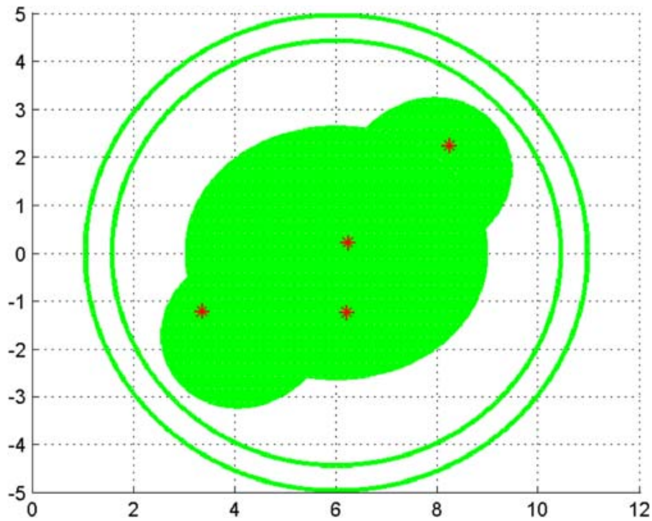

Figure 5. $\Omega(T) \subseteq \mathcal{K}(T) \subseteq \Gamma(T)$.

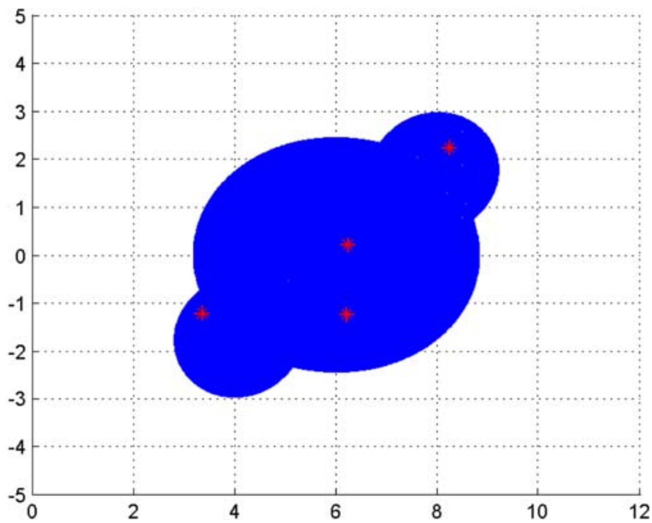

Figure 6. $\Omega^1(T)$.

where $n \geq 5$. From Theorem 3.2 (Theorem 3.1), we get that $\Omega^2(T)$ ($\Omega^1(T)$) is constant as n grows. Note that $\sigma(T)$ changes when n changes. However, $\Omega^2(T)$ ($\Omega^1(T)$) always contains all eigenvalues of T. This is shown by Figure 8 for the cases $n = 20, 50, 200$. Hence, the new regions $\Omega^1(T)$ and $\Omega^2(T)$ capture effectively all eigenvalues of Toeplitz matrices with a higher order.

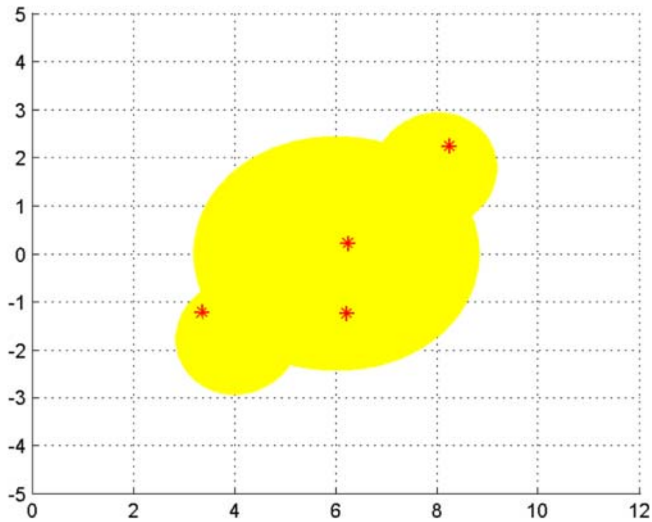

Figure 7. $\Omega^2(T)$.

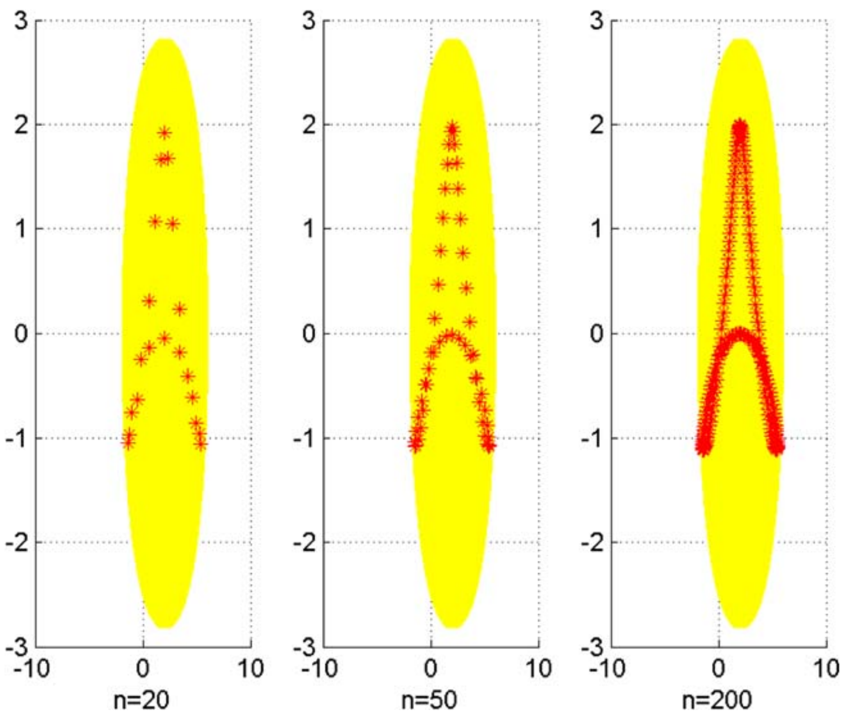

Figure 8. $\Omega^2(T)$ for n =20, 50, 200.

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