Adv Comput Math (2015) 41:55–75
DOI 10.1007/s10444-014-9349-0

On general principles of eigenvalue localizations via diagonal dominance

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Received: 9 August 2013 / Accepted: 13 March 2014 /
Published online: 28 April 2014

Abstract This paper suggests a unifying framework for matrix spectra localizations that originate from different generalizations of strictly diagonally dominant matrices. Although a lot of results of this kind have been published over the years, in many papers same properties were proven for every specific localization area using basically the same techniques. For that reason, here, we introduce a concept of DD-type classes of matrices and show how to construct eigenvalue localization sets. For such sets we then prove some general principles and obtain as corollaries many singular results that occur in the literature. Moreover, obtained principles can be used to construct and use novel Geršgorin-like localization areas. To illustrate this, we first prove a new nonsingularity result and then use established principles to obtain the corresponding localization set and its several properties. In addition, some new results on eigenvalue separation lines and upper bounds for spectral radius are obtained, too.

Keywords Eigenvalues · Diagonal dominance · Geršgorin's set · H-matrices

Mathematics Subject Classifications (2010) 65F15 · 15A18 · 15A22

1 Introduction and motivations

Since Geršgorin's theorem was published in 1931, there has been many results about matrix eigenvalue localization of the similar type, i.e., by the use of analytic curves in the complex plane obtained via absolute row/column sums (or their parts) in different

Communicated by: R. Bru

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combinations - sums, products, convex combinations, etc. Practically all of them are related to a specific class of nonsingular matrices, sometimes published in the same paper, and sometimes obtained by different authors. An extensive survey of many of them can be found in the book of R. S. Varga, Geršgorin and his circles [50], and for some of the latter advances see [8, 13, 14, 16, 24–26, 33, 35, 37, 42]. Among many of these results one can notice several interesting properties:

- eigenvalues are localized by a compact set in a complex plane,
- if a matrix class contains all strictly diagonally dominant matrices, the related localization set is inside Geršgorin's set,
- nonsingularity of matrices in a certain class is equivalent to the fact that the related localization set containes the spectra for an arbitrary matrix,
- if a localization set consists of several disjoint parts, then the number of eigenvalues in each part is known,
- by inspecting the distribution of diagonal entries in the complex plane, for a specific class of matrices, inertia of a matrix can be obtained,
- by analysing localization set, a bound for spectral radius of an arbitrary matrix can be obtained.

While the first two items are clear in all the papers, the third one was for the first time explicitly stated in [50] where it was used as a recurring theme. The forth item was proved for some localization sets, and the two last items were just implicitly used in many applications concerning stability issues in engineering, for example see [6, 7, 17, 19, 21, 34, 43, 44].

As we shall see, these facts are implied by principles that are obtained from only few general properties of a matrix class in consideration. Following the order of the items in the list above, we will name them:

- Compactness principle,
- Monotonicity principle,
- Equivalence principle,
- Isolation principle,
- Inertia principle,
- Spectral radius principle

The paper is organised as follows. In the second section we define the concept of DD-type classes of matrices that will be used as a tool in the later sections. Then, we establish that the maximal nonsingular DD-type class is the class of nonsingular H-matrices. Third section starts with the Equivalence principle for eigenvalue localizations, which is used to clearly define what Geršgorin-type localization sets are. This term is often used in the literature, but not precisely defined up to now. Then, Monotonicity, Compactness, Isolation, Inertia and Spectral radius principles are established, and their usage is illustrated by introducing some new classes of matrices and their corresponding eigenvalue localization sets. As a consequence, section 3 also contains some new easily computable upper bounds for spectral radius of an arbitrary matrix.

2 DD-type classes of matrices

Throughout this paper, for an arbitrary n from the set of positive integers $\mathbb{N}$, by $\mathbb{C}^n$ we denote a complex n-dimensional vector space of column vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, where $x_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, and, for arbitrary $m, n \in \mathbb{N}$, by $\mathbb{C}^{m,n}$ we denote the collection of all $m \times n$ matrices with complex entries. In a similar way, by $\mathbb{R}^n$ and $\mathbb{R}^{m,n}$, we denote, respectively, the real n-dimensional vector space of vector columns, and the collection of rectangular matrices with real entries. A matrix $A \in \mathbb{C}^{m,n}$ with entries $a_{ij} := (A)_{ij} \in \mathbb{C}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$ is denoted by $A = [a_{ij}]$, the identity matrix by I and zero matrix by O.

Throughout the paper, for any two real matrices $A = [a_{ij}] \in \mathbb{R}^{m,n}$ and $B = [b_{ij}] \in \mathbb{R}^{m,n}$, $A \geq B$ is understood entry-wise, i.e., $A \geq B$ if and only if $a_{ij} \geq b_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. In the same way we use notations $A \leq B$, $A > B$, $A < B$.

Finally, the set of indices is denoted by $N := \{1, 2, \ldots, n\}$, and

$$r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{ij}| \quad (i \in N) \tag{2.1}$$

is the sum of moduli of **i-th deleted row** of the matrix A. If matrix A is of the dimension $n = 1$, we define $r_1(A) := 0$.

In this notation, the famous result, which appeared in the early paper of Lévy in 1881, [32], and then latter, independently, in the work of Minkowski in 1900, [36], (real case), the paper of Desplanque in 1887, [18], and the book of Hadamard in 1903, [23] (complex case), is that $|a_{ii}| > r_i(A)$, for all $i \in N$ implies nonsingularity of the given square complex matrix A. For obvious reasons, the class of all such matrices is called **strictly diagonally dominant (SDD)** class. Its strong connection to the class of well known M-matrices, and more general H-matrices, is well established in the work of many authors (see [3] for extensive survey of this relationship). Here we recall just few important facts.

While SDD matrices have many generalizations, the term *generalized diagonal dominance* has a precise sense. It dates back to the early seventies, when the convergence theory of iterative methods was highly attractive area of research. In the work of James and Riha from 1974, [30] a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is said to be **generalized diagonally dominant (GDD)** if there exists an entry-wise positive vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, such that

$$|a_{ii}|x_i > \sum_{j \in N \setminus \{i\}} |a_{ij}|x_j \quad (i \in N), \tag{2.2}$$

or, equivalently, AX is an SDD matrix, where $X := \operatorname{diag} x_1, x_2, \ldots, x_n$.

Obviously, all GDD matrices (obtained by various scalings of column vectors of SDD matrices) are nonsingular, and, although they appear as a fairly easy and simple observation, it turned out that these matrices can be a very useful tool, mainly through their connection to theory of M-matrices.

The name M-matrices originates form the work of Ostrowski in 1937, who, starting from the already mentioned result of Minkowski on nonsingularity, [36], referred to a certain class of matrices as Minkowski-matrices, or **M-matrices**. Since then, more than seventy different equivalent definitions of M-matrices were discovered, and many famous mathematicians have given their contributions in this direction. Thus, connections with the Perron-Frobenius theory of nonnegative matrices, positive definiteness, positive stability and diagonal dominance are just some of them. Here, we will use one of the basic definitions of M-Matrices, condition (N_{38}) of Theorem 6.2.3 in [3].

A real matrix $A \in \mathbb{R}^{n,n}$ is called a **nonsingular M-matrix** if all of the following conditions hold:

1. $a_{ii} > 0$, for all $i \in N$,
2. $a_{ij} \leq 0$, for all $i, j \in N, i \neq j$,
3. A is nonsingular, i.e., A^{-1} exists, and
4. A is inverse (entry-wise) nonnegative, i.e., $A^{-1} \geq O$.

Actually, this class of matrices come directly from economic models in the form of the well-known Hawkins-Simon condition, [3]. So, from the early beginnings, up to now, this theory has been doubly motivated and conducted. On one hand, mathematicians developed a wide range of applications in establishing bounds on the eigenvalues of nonnegative matrices, establishing convergence criteria for iterative methods for solving large sparse systems of linear equations, localizing eigenvalues, while on the other hand, economy researchers have studied gross substitutability, stability of general equilibrium and Leontief's input/output analysis of economic systems. If one goes thoroughly through contemporary research in mathematics, engineering, robotics, ecology, pharmaceutical modelling, economics, and many others, one can find all sorts of applications of the M-matrix theory.

How this definition is related to GDD matrices, for the first time was published in a famous paper of Fiedler and Pták in 1962, [20]. It was proved that a matrix that fulfils conditions (1.) and (2.) given above is an M-matrix if and only if it is generalised diagonally dominant. Thus, a natural extension to the complex case followed.

Given an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, its **comparison matrix** $\langle A \rangle := [m_{ij}] \in \mathbb{R}^{n,n}$ is defined by

$$m_{ij} := \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & \text{otherwise.} \end{cases} \tag{2.3}$$

A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called a **nonsingular H-matrix** if its comparison matrix is a nonsingular M-matrix, i.e. if $\langle A \rangle$ is nonsingular, and $\langle A \rangle^{-1} \geq O$.

Thus, from the result of Fiedler and Pták in 1962, [20], we have one of the key theorems in the theory of H-matrices.

Theorem 1 *An arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a nonsingular H-matrix if and only if it is a generalized diagonally dominant matrix, i.e., there exists a positive diagonal matrix X, such that AX is a strictly diagonally dominant matrix.*

Therefore, the classes of nonsingular H-matrices and GDD matrices are the same, and, in the following, we will denote them by $\mathbb{H}$.

Having this in mind, natural questions arise: What other generalizations of SDD matrices are (not) GDD matrices?

In the literature, there are different generalizations of SDD matrices. For example, the class of :

- irreducibly diagonally dominant matrices from [45],
- semi-strictly diagonally dominant matrices from [2],
- doubly strictly/irreducibly diagonally dominant matrices from [38],
- Brualdi strictly diagonally dominant matrices and Brualdi irreducibly diagonally dominant matrices from [5],
- S-strictly diagonally dominant matrices and S-irreducibly diagonally dominant matrices from [9, 15],
- Ostrowski strictly diagonally dominant matrices of type I and type II from [4, 39],

are just some of them. For all of these classes it is known, implicitly or explicitly, that they do belong to $\mathbb{H}$, and for almost all of them this fact had to be proved separately from the nonsinularity results (provided in cited references). So, naturally, we are interested to find a *unifying framework* for all of them. Namely, one can ask: What else do we need, apart from nonsingularity of all the matrices in a class (an obvious necessary condition), in order to conclude that *all* mentioned classes are contained in the class of nonsingular H-matrices? To that end, we introduce the concept of DD-type matrices.

Definition 1 Let $\mathbb{K}$ be a nonempty class of square matrices of an arbitrary size. If $\mathbb{K}$ is such that:

- for any $A \in \mathbb{K}$, diagonal entries of A are nonzero,
- for any $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $A \in \mathbb{K}$ if and only if $|A| \in \mathbb{K}$, where $|A| := [|a_{ij}|]$,
- for every $A \in \mathbb{K}$ and every $B \in \mathbb{C}^{n,n}$, if $\langle B \rangle \geq \langle A \rangle$, then $B \in \mathbb{K}$,

then we say that $\mathbb{K}$ is a **diagonally dominant-type**, or briefly **DD-type**, class of matrices.

In other words, a matrix class is a DD-type class if all its matrices have nonzero diagonal, the class is invariant under growth of moduli of diagonal entries, decay of moduli of off-diagonal entries, and the arbitrary change of complex argument of an arbitrary entry.

Therefore, since one can easily check that all the above mentioned classes are DD-type classes, we obtain that they are all subclasses of $\mathbb{H}$ directly from the corresponding nonsingularity results by the following theorem.

Theorem 2 *If a diagonally dominant-type class of matrices $\mathbb{K}$ is a class of nonsingular matrices, then it is a subclass of nonsingular H-matrices, i.e.,* $\mathbb{K} \subseteq \mathbb{H}$.

Proof Take an arbitrary $A \in \mathbb{K}$. Since $|\langle A \rangle| = |A| \in \mathbb{K}$, we have that $\langle A \rangle \in \mathbb{K}$, hence, $\langle A \rangle$ is nonsingular. We need to prove that $\langle A \rangle^{-1}$ is nonnegative. Take a splitting of $\langle A\rangle = D_A - B_A$, where $D_A := \mathrm{diag}\langle A\rangle = \mathrm{diag}(|a_{11}|, |a_{22}|, \ldots, |a_{nn}|)$. Obviously, D_A is a diagonal matrix with positive diagonal entries, so, we can write $\langle A\rangle = D_A(I - D_A^{-1}B_A)$, which implies that $I - D_A^{-1}B_A$ is nonsingular, and $\langle A\rangle^{-1} = (I - D_A^{-1}B_A)^{-1}D_A^{-1}$.

Let us show that $\rho(D_A^{-1}B_A) := \max\left\{|\lambda| : \lambda \in \sigma(D_A^{-1}B_A)\right\} < 1$. Assume, on the contrary, that there exists $\lambda \in \sigma(D_A^{-1}B_A)$, such that $|\lambda| \geq 1$. Then, $\lambda I - D_A^{-1}B_A = D_A^{-1}(\lambda D_A - B_A)$ is singular. But, since $|\lambda| \geq 1$, and $D_A, B_A \geq 0$ we can write $|\lambda D_A - B_A| = |\lambda|D_A + B_A = D_A + B_A + (|\lambda| - 1)D_A = |A| + D$, where $D := (|\lambda| - 1)D_A$ is nonnegative diagonal matrix. Hence, $\lambda D_A - B_A \in \mathbb{K}$, and, therefore, nonsingular, which is an obvious contradiction.

Now, since, $\rho(D_A^{-1}B_A) < 1$, geometric series $\sum_{k=0}^{\infty}(D_A^{-1}B_A)^k$ converges to $(I - D_A^{-1}B_A)^{-1}$. Having that $D_A^{-1}B_A$ is nonnegative, the limit of the series is nonnegative, which completes the proof. □

Since $\mathbb{H}$ is a DD-type, too, $\mathbb{H}$ can be considered as the maximal nonsingular class of the DD-type.

Of course, above mentioned matrix classes make just a short list of many different nonsingularity results to which Theorem 2 can be applied. Therefore, definition of DD-type classes is a wide framework for considering properties of nonsingular matrix classes and their related eigenvalue localization sets.

To end this section, we prove a new nonsingularity result that defines a DD-type class of matrices. This new result is motivated by the fact that sometimes a concrete problem from applications constrains one to use a specific norm on the $\mathbb{C}^n$. So, in such circumstances, using an infinity norm (closely related to SDD matrices) may be inappropriate, and one may wish to use some other p-norm instead. To that end, we introduce a class of matrices called SDD(p) that is constructed using p-norm and its dual q-norm. We note that a special case when $p = q = 2$ can be found in [29], and that the theorem we present here can be easily generalised to infinite matrices on ℓ^p spaces, where its meaningfulness is much more justified.

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and an arbitrary $p \in [1, \infty]$, with

$$r_i^p(A) := \Big(\sum_{j\in N\setminus\{i\}} |a_{ij}|^p\Big)^{\frac{1}{p}} \quad (i \in N) \tag{2.4}$$

denote a p-norm of the i-th deleted row of the matrix A. So, following the prior considerations, one may ask if $|a_{ii}| > r_i^p(A)$, for all $i \in N$, implies nonsingularity of the given matrix A. Certainly, this is the case when $p = 1$ (SDD matrices), but in general this is not sufficient as a simple example shows.

Example 3 Let

$$M_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Then, since $M_1 e = 0$, where $e = [1\ 1\ 1]^T$, M_1 is a singular matrix. But, for every $p > 1$, $2 > \sqrt[p]{2}$, and, consequently, matrix M_1 fulfils $|(M_1)_{ii}| > r_i^p(M_1)$, for all $i \in N$.

Therefore, something needs to be added in order to obtain nonsingularity result for an arbitrary p-norm. The following theorem provides an answer to that.

Theorem 4 *Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and an arbitrary $p \in [1, \infty]$, if there exists an entry-wise positive vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T > 0$ such that $||\mathbf{x}||_q \leq 1$, where q is Hölder's complement of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, so that*

$$x_i |a_{ii}| > r_i^p(A), \quad \text{for all } \ i \in N, \tag{2.5}$$

then, A is nonsingular.

Proof Suppose that the matrix A is singular, i.e., that there exists $y \in \mathbb{C}^n \setminus \{0\}$, such that $A\mathbf{y} = 0$. Since A is linear, without the loss of generality, we can assume that $||\mathbf{y}||_q = 1$.

Observe that $A\mathbf{y} = 0$ is equivalent to

$$a_{ii} y_i = - \sum_{j \in N \setminus \{i\}} a_{ij} y_j \quad (i \in N)$$

which implies that

$$|a_{ii}||y_i| = |\sum_{j \in N \setminus \{i\}} a_{ij} y_j| \quad (i \in N) \tag{2.6}$$

Then, apply Hölder's inequality to the right hand side of Eq. 2.6 and obtain

$$|a_{ii}||y_i| \leq r_i^p(A)||y||_q = r_i^p(A) \quad (i \in N)$$

But, these inequalities coupled with Eq. 2.5 imply that $x_i > |y_i|$, for all $i \in N$. Thus, $||\mathbf{x}||_q > ||\mathbf{y}||_q = 1$, and the proof is completed. □

Given a fixed value $p \in [1, \infty]$, the class of nonsingular matrices from the previous theorem we denote as **SDD(p) class**. These SDD(p) classes of matrices can also be characterised in the following way.

Proposition 5 *Given an arbitrary $p \in [1, \infty]$, matrix A is an SDD(p) matrix if and only if*

$$||\delta_p(A)||_q < 1, \tag{2.7}$$

where $\delta_p(A) := [\delta_1, \delta_2, \ldots, \delta_n]^T$, $\delta_i := \frac{r_i^p(A)}{|a_{ii}|}$, for $i \in N$, and q is Hölder's complement of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Obviously, if A is an SDD(p) matrix, then there exist $\mathbf{x} > 0$ such that $||\mathbf{x}||_q \leq 1$ and $\delta_p(A) < \mathbf{x}$, implying that $||\delta_p(A)||_q < 1$. On the other hand, if $||\delta_p(A)||_q < 1$, then there exist a sufficiently small $\varepsilon > 0$ such that for $x_i := \delta_i + \varepsilon > 0$, where $i \in N$, $||\mathbf{x}||_q < 1$. But, then, for the vector $\mathbf{x}$ Eq. 2.5 holds, and A is an SDD(p) matrix. □

Taking $p = 1$, from the previous corollary, A is an SDD(1) if and only if

$$\max_{i \in N} \frac{r_i(A)}{|a_{ii}|} < 1,$$

which is equivalent to the fact that A is an SDD matrix. The other extreme case is SDD(∞) class, i.e., class of matrices A such that

$$\frac{\max_{j \neq i} |a_{1j}|}{|a_{11}|} + \frac{\max_{j \neq i} |a_{2j}|}{|a_{22}|} + \ldots + \frac{\max_{j \neq i} |a_{nj}|}{|a_{nn}|} < 1.$$

It is important to note here that, while in general, for $p_1 > p_2 \geq 1$, $r_i^{p_1}(A) \leq r_i^{p_2}(A)$, for every A, and all $i \in N$, classes SDD(p_1) and SDD(p_2) stand *in a general position*. This is due to the scaling vector which is bounded by one in a dual norm of a given p-norm.

Furthermore, by inspection of Eq. 2.5, we immediately conclude that SDD(p) is a DD-type class, and, thus, as a consequence of Theorem 2, it is a subclass of $\mathbb{H}$. Therefore, we have for each value of p a different subclass of the class of nonsingular H-matrices.

3 Principles of Geršgorin-type eigenvalue localizations

Given an arbitrary square matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, the set of its eigenvalues is called the **spectrum**, and is denoted by $\sigma(A)$, i.e.,

$$\sigma(A) := \{\lambda \in \mathbb{C} : det(\lambda I - A) = 0\}. \tag{3.1}$$

Additionally, we define

$$\begin{cases} \Gamma_i(A) := \{z \in \mathbb{C} \ : \ |z - a_{ii}| \leq r_i(A)\}, \ (i \in N), \\ \Gamma(A) := \bigcup_{i \in N} \Gamma_i(A). \end{cases} \tag{3.2}$$

The following proposition is the famous Geršgorin's theorem [22]. In order to be in the agreement with contemporary notations, we give it in the form as it was stated in Theorem 1.1 of [50].

Theorem 6 (Geršgorin's first theorem) *Given an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, let λ be an eigenvalue. Then, there exists an index $k \in N$, such that*

$$|\lambda - a_{kk}| \leq r_k(A), \tag{3.3}$$

implying that $\lambda \in \Gamma_k(A)$, and, therefore $\lambda \in \Gamma(A)$. Since $\lambda \in \sigma(A)$ is arbitrary, consequently, it follows that

$$\sigma(A) \subseteq \Gamma(A). \tag{3.4}$$

Observe that the set $\Gamma_i(A)$, which we will call the ***i*-th Geršgorin disk** of the matrix A, is the closed disk in a complex plane, centered in a_{ii}, and with radius $r_i(A)$. The set $\Gamma(A)$ is, therefore, the union of all Geršgorin disks, and, consequently, it is closed and bounded, i.e., it is a compact subset of $\mathbb{C}$, which contains the spectrum of the matrix A. For the diversity of geometrical structures of Geršgorin sets and the various ways how it captures eigenvalues, one can see [50]. One very well known fact about this is the second result of Geršgorin in the same paper from 1931, [22], which gives the possibility to isolate an eigenvalue if one succeeds to make a Geršgorin disk disjoint from all the others.

Given $n \geq 2$ and $S \subseteq N$, by $|S|$, we denote the **cardinality** of the set S, i.e., the number of its elements, and by $\overline{S} := N \setminus S$ its complement. Furthermore, given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $\Gamma_S(A) := \bigcup_{i \in S} \Gamma_i(A)$ denotes the part of Geršgorin set that "corresponds" to the indices from the set S.

Theorem 7 (Geršgorin's second theorem) *Given an arbitrary matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *and set of indices* $S \subsetneq N$, *if*

$$\Gamma_S(A) \bigcap \Gamma_{\overline{S}}(A) = \emptyset, \tag{3.5}$$

then $\Gamma_S(A)$ *contains exactly* $|S|$ *eigenvalues of the matrix* A, *and, consequently,* $\Gamma_{\overline{S}}(A)$ *contains the reminder of the spectrum of* A.

Observing Eq. 3.2, one could notice that the fact $0 \notin \Gamma(A)$ is equivalent to the fact that A is an SDD matrix, implying the Lévy-Desplanques theorem. In fact, the relationship between these two famous theorems is very strong. More precisely, there exists an ***equivalence*** between the Geršgorin's theorem and the Lévy-Deslanques's theorem, which is high-lighted in the book of Varga [50].

In other words, starting from the fact that Geršgorin's theorem holds, we can show that every SDD matrix is nonsingular, i.e., the theorem of Lévy-Desplanques holds. And, vice versa, assuming that every SDD matrix is nonsingular, we can deduce that all eigenvalues of an arbitrary matrix are inside of the Geršgorin's set.

In fact, this simple reasoning leads to the construction of general eigenvalue localization sets.

Theorem 8 (Equivalence Principle) *Given a class of square complex matrices of an arbitrary size, denoted by* $\mathbb{K}$, *for an arbitrary square matrix* A, *define the set of complex numbers*

$$\Theta^{\mathbb{K}}(A) := \{z \in \mathbb{C} \ : \ zI - A \notin \mathbb{K}\}\,. \tag{3.6}$$

Then, the following two conditions are equivalent:

- *All matrices from* $\mathbb{K}$ *are nonsingular;*
- *Given an arbitrary square matrix* A, *the set* $\Theta^{\mathbb{K}}(A)$ *contains all its eigenvalues, i.e.,* $\sigma(A) \subseteq \Theta^{\mathbb{K}}(A)$.

Proof Assume that all matrices in $\mathbb{K}$ are nonsingular. Taking an arbitrary matrix $A \in \mathbb{C}^{n,n}$ and $\lambda \in \sigma(A)$, the matrix $\lambda I - A$ is singular, and, hence, $\lambda I - A \notin \mathbb{K}$. Therefore, $\lambda \in \Theta^{\mathbb{K}}(A)$, and, consequently, $\sigma(A) \subseteq \Theta^{\mathbb{K}}(A)$.

To prove the opposite implication, assume that for every $A \in \mathbb{C}^{n,n}$, $\sigma(A) \subseteq \Theta^{\mathbb{K}}(A)$. Now, assume that $A \in \mathbb{K}$ is singular. Then, $0 \in \sigma(-A)$, and, consequently, $0 \in \Theta^{\mathbb{K}}(-A)$. But, this is equivalent to the fact that $0I - A = -A \notin \mathbb{K}$, which is an obvious contradiction. Thus, every $A \in \mathbb{K}$ is nonsingular. $\square$

While, in the extreme case, when $\mathbb{K}$ is taken to be the class of *all* nonsingular matrices, $\Theta^{\mathbb{K}}(A) = \sigma(A)$ holds for every A, by narrowing the class $\mathbb{K}$, we are obtaining the sets $\Theta^{\mathbb{K}}(A)$ which become, in general, "wider", and, thus, we are obtaining an "approximation" of the spectrum. In other words, we get a certain localization set for the spectrum.

But, the question is how "interesting" the obtained localization set is? In other words, are we able to "easily" construct it in the complex plane, and is the cost of this significantly less then calculating the actual eigenvalues themselves?

In the case of Geršgorin's theorem, where $\Theta^{\mathbb{K}}(A) = \Gamma(A)$, we have seen that the class $\mathbb{K}$ is the class of all SDD matrices. Therefore, somewhat naturally, the set $\Theta^{\mathbb{K}}(A) = \{z \in \mathbb{C} : zI - A \notin \mathbb{K}\}$ we will call **Geršgorin-type set**, if $\mathbb{K}$ is a diagonally dominant-type class of nonsingular matrices.

In another words, due to Theorem 2, Geršgorin-type localization sets are constructed from subclasses of nonsingular H-matrices.

Of course, an interesting question is what Geršgorin-type localization set corresponds to the maximal DD-type nonsingular class of matrices? The answer is related to the first generalization of Geršgorin's theorem that was considered in his original paper in 1931.

Taking an arbitrary $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and an entry-wise positive vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, the matrix $X^{-1}AX = [\frac{a_{ij}x_j}{x_j}]$, where $X :=$ diag $(x_1, x_2, \ldots, x_n)$, has the same spectrum as A, while their Geršgorin sets can significantly differ. So, in order to localize eigenvalues of the matrix A, we can apply Geršgorin's theorem to the matrix $X^{-1}AX$. Then, we have n positive parameters which we can arbitrarily choose, and, hence, influence the shape and the size of the localization set. The minimal localization set obtained in this way for a given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$:

$$\Gamma^{\mathfrak{R}}(A) := \bigcap_{\mathbf{x}>\mathbf{0}} \Gamma(X^{-1}AX), \tag{3.7}$$

is known as the **Minimal Geršgorin set** of the matrix A. Obviously, according to the Theorem 2, it is the best possible Geršgorin-type set, i.e. $\Gamma^{\mathfrak{R}}(A) = \Theta^{\mathbb{H}}(A)$ holds for every matrix A.

This set was introduced in [47], its theoretical properties were studied in [31, 48–50] and an iterative algorithm for its approximation was developed in [51]. As it turns out, to obtain this set for a given matrix is not an easy task. Therefore, many authors were motivated to investigate localization sets that are giving less precise results, but are much easier to handle.

Moreover, when constructing such localization sets a certain monotonicity can be observed. Namely, given two nonsingular classes of matrices $\mathbb{K}_1$ and $\mathbb{K}_2$, by definition in Eq. 3.6, $\mathbb{K}_1 \subseteq \mathbb{K}_2$ if and only if $\Theta^{\mathbb{K}_2}(A) \subseteq \Theta^{\mathbb{K}_1}(A)$ holds for every matrix $A \in \mathbb{C}^{n,n}$. To this behavior we will simply refer as **Monotonicity principle**. It simply states that when the class of nonsingular matrices gets wider, the corresponding eigenvalue localization set gets smaller.

In order to generalise property of compactness and the result of Theorem 7 to all such cases of Geršgorin-type localization sets we introduce Compactness and Isolation principle.

Theorem 9 (Isolation Principle) *Given a nonsingular DD-type class of matrices $\mathbb{K}$ and arbitrary matrix $A \in \mathbb{C}^{n,n}$, $n \geq 2$, if there exist closed disjoint sets $U, V \subseteq \mathbb{C}$ such that for the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$*

$$\Theta^{\mathbb{K}}(A) = U \cup V, \tag{3.8}$$

then, the number of eigenvalues and the number of diagonal entries of the matrix A in the set U coincide.

Proof Let $D_A := \mathrm{diag}\,(a_{11}, a_{22}, \ldots, a_{nn})$. Take the splitting of the matrix $A = D_A - F_A$, and consider the family of matrices $A(t) := D_A - tF_A$, for $0 \leq t \leq 1$.

First, we prove that $\Theta^{\mathbb{K}}(A(t)) \subseteq \Theta^{\mathbb{K}}(A)$ for $t \in [0,\ 1]$. Let $t \in [0,\ 1]$ and take $z \in \Theta^{\mathbb{K}}(A(t))$. Then, $zI - A(t) \notin \mathbb{K}$, and, since

$$\langle zI - A(t)\rangle = |zI - D_A| - t|F_A| \geq |zI - D_A| - |F_A| = \langle zI - A\rangle,$$

due to the fact that since $\mathbb{K}$ is DD-type class, $zI - A \notin \mathbb{K}$, too. Therefore, $\Theta^{\mathbb{K}}(A(t)) \subseteq \Theta^{\mathbb{K}}(A)$, for all $t \in [0,\ 1]$.

Let us now consider the case when $t = 0$. Then, $A(0) = D_A$, and $z \in \Theta^{\mathbb{K}}(A(0))$ if and only if $zI - D_A \notin \mathbb{K}$. Obviously, if $z = a_{ii}$, for some $i \in N$, then $zI - D_A$ has a zero on diagonal. Thus it can not be in $\mathbb{K}$ which is the SDD-type class of matrices. Therefore, $a_{ii} \in \Theta^{\mathbb{K}}(A(0))$, for all $i \in N$. For the same reasons, $a_{ii} \in \Theta^{\mathbb{K}}(A)$, $i \in N$. On the other hand, when $z \neq a_{ii}$, for all $i \in N$, $zI - D_A$ is a nonsingular diagonal matrix, implying that $zI - D_A \in \mathbb{K}$, i.e., $z \notin \Theta^{\mathbb{K}}(A(0))$. So, $\Theta^{\mathbb{K}}(A(0)) = \{a_{11},\ a_{22}, \ldots, a_{nn}\}$. In another words, we have obtained that $\Theta^{\mathbb{K}}(A(0)) = \sigma(A(0)) = \{a_{11},\ a_{22}, \ldots, a_{nn}\}$, and that $\Theta^{\mathbb{K}}(A(t)) \subseteq \Theta^{\mathbb{K}}(A)$, for all $t \in [0,\ 1]$.

Now, since the sets $U, V \subseteq \mathbb{C}$ are disjoint, and $\{a_{11},\ a_{22}, \ldots, a_{nn}\} \subseteq \Theta^{\mathbb{K}}(A) = U \cup V$, if the number of diagonal entries that lie in the set U is denoted by m, then, of course, $n - m$ diagonal entries of the matrix $A(0)$ lie in the set V.

Let us with $\lambda(t)$ denote the eigenvalue of the matrix $A(t)$ that, for $t = 0$, becomes a diagonal entry that lies in the set $U \subseteq \mathbb{C}$. Then, for all $t \in [0,\ 1]$, we have that $\lambda(t) \in \Theta^{\mathbb{K}}(A(t)) \subseteq \Theta^{\mathbb{K}}(A)$, and since the eigenvalues are continuous functions of matrix entries, [40], we can consider $\{\lambda(t)\ :\ t \in [0,\ 1]\}$, as a continuous curve in the complex plane such that $\{\lambda(t)\ :\ t \in [0,\ 1]\} \subseteq \Theta^{\mathbb{K}}(A) = U \cup V$.

But, sets U and V are disjoint and closed parts of the compact set in $\mathbb{C}$, so, if $\lambda(1) \in V$, then the continuous curve has a part out of $U \cup V$. Therefore, since $\lambda(0) \in U$, so is $\lambda(1) \in U$. Consequently, the number of eigenvalues of $A(1) = A$ that are in the set U is m. □

Again, taking any known DD-type class of nonsingular matrices (for example doubly strictly diagonally dominant matrices, Brualdi strictly diagonally dominant matrices, Ostrowski strictly diagonally dominant matrices of type I and type II, etc.) by Equivalence Principle we obtain the corresponding localization sets (Brauer's ovals of Cassini, Brualdi lemniscate sets, α_1 and α_2 localization sets, etc., respectively), and, by Isolation Principle, we are able to determine the number of eigenvalues in their disjoint parts. So, the known theorems on the number of eigenvalues in disjoint parts of these Geršgorin-type localization sets, [50], are, in fact, corollaries of the presented Isolation Principle, and, up to the author's knowledge, the result of this kind hasn't been stated in such a general way with a wide applicability.

In the following, we will need two more properties of matrix classes that come naturally with many generalizations of strict diagonal dominance. Namely, for a given class of matrices $\mathbb{K}$ is we say that it is:

- **star-shaped**, if for every real $\alpha > 0$, $A \in \mathbb{K}$ implies $\alpha A \in \mathbb{K}$.
- **open**, if for every matrix $A \in \mathbb{K}$, there exists an arbitrary small $\varepsilon > 0$, such that for every matrix $B \in \mathbb{C}^{n,n}$, $|(A - B)_{ij}| < \varepsilon$, for all $i, j \in N$, implies $B \in \mathbb{K}$.
- **SDD-type** class if it is open star-shaped DD-type class of matrices and $I \in \mathbb{K}$.

Here, note that, since $\mathbb{H}$ is also SDD-type class, it is the maximal SDD-type class due to Theorem 2.

Theorem 10 (Compactness Principle) *Given a nonsingular SDD-type class of matrices $\mathbb{K}$ and an arbitrary matrix $A \in \mathbb{C}^{n,n}$, the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$ is a compact set in complex plane.*

Proof Given an arbitrary $A \in \mathbb{C}^{n,n}$, we first observe that $\Theta^{\mathbb{K}}(A)$ is a closed set as a consequence of the fact that $\mathbb{K}$ is an open class. Namely, for every $z \in \mathbb{C} \setminus \Theta^{\mathbb{K}}(A)$, $zI - A \in \mathbb{K}$, so, there exists a small $\varepsilon > 0$ such that for every $\omega \in \mathbb{C}$, $|z - \omega| < \varepsilon$ implies $(\omega I - A) \in \mathbb{K}$, i.e., $\omega \in \mathbb{C} \setminus \Theta^{\mathbb{K}}(A)$. Therefore, $C \setminus \Theta^{\mathbb{K}}(A)$ is an open set in $\mathbb{C}$, implying that $\Theta^{\mathbb{K}}(A)$ is closed.

Next, suppose that $\Theta^{\mathbb{K}}(A)$ is unbounded. Then, there exists a sequence of complex numbers $\{z_k\}_{k\in\mathbb{N}} \subseteq \Theta^{\mathbb{K}}(A)$ such that $|z_k| \to \infty$, when $k \to \infty$. For sufficiently large $k \in \mathbb{N}$ we consider the matrix $M_k := I - (z_k)^{-1}A$. Obviously, $|z_k||M_k| = |z_k I - A| \notin \mathbb{K}$, which implies that $M_k \notin \mathbb{K}$. But, since $I \in \mathbb{K}$, there exists small $\varepsilon > 0$, such that for every $B \in \mathbb{C}^{n,n}$, $|(I - B)_{ij}| < \varepsilon$, for all $i, j \in N$, implies $B \in \mathbb{K}$. So, for a $k \in \mathbb{N}$ such that $|z_k| > \max_{i,j\in N} |a_{ij}|\varepsilon^{-1}$, $|(I - M_k)_{ij}| < \varepsilon$, for all $i, j \in N$, implying that $M_k \in \mathbb{K}$. Therefore, set $\Theta^{\mathbb{K}}(A)$ has to be bounded. $\square$

In order to illustrate the principles established so far, we can start with our new classes of SDD(p) matrices, and, using the Equivalence principle, obtain the corresponding Geršgorin-type sets.

Given an arbitrary $p \in [1, \infty]$, let q be Hölder's complement of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, and let $\mathbb{S}_p$ denote the class of all SDD(p) matrices. Then, for an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ the corresponding Geršgorin-type set $\Theta^{\mathbb{S}_p}(A)$ can be expressed as

$$\Theta^{\mathbb{S}_p}(A) = \bigcap_{||\mathbf{x}||_q \le 1} \bigcup_{i\in N} \Gamma_i^{p,\mathbf{x}}(A),$$

where $\Gamma_i^{p,\mathbf{x}}(A) := \left\{z \in \mathbb{C} \ : \ x_i|z - a_{ii}| \le r_i^p(A)\right\}$, for all $i \in N$.

In case when $p = 1$, obviously, $\Theta^{\mathbb{S}_p}(A) = \Gamma(A)$. But, in case when $p > 1$ this set can be expressed, equivalently, as

$$\Theta^{\mathbb{S}_p}(A) = \left\{ z \in \mathbb{C} \ : \ \sum_{i\in N} \left[\frac{r_i^p(A)}{|z - a_{ii}|} \right]^{\frac{p}{p-1}} \ge 1 \right\}.$$

First, since the classes $\mathbb{S}_p$ for different values of $p \in [1, \infty]$ stand in a general position to one another, according to the Monotonicity principle, the same holds for the sets $\Theta^{\mathbb{S}_p}(A)$, provided a *generic* matrix A.

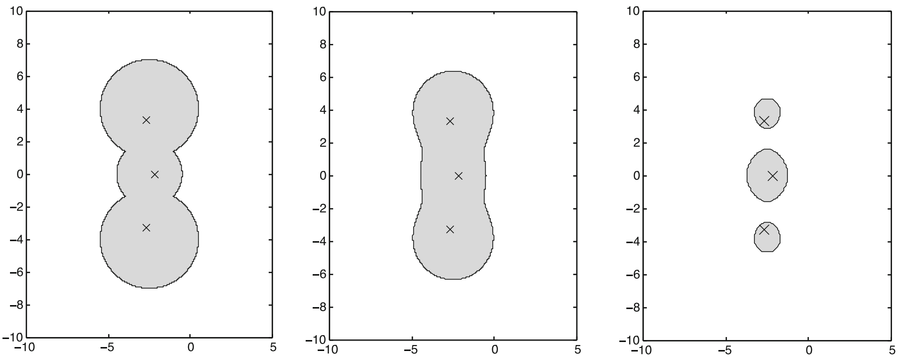
Second, since by inspection of Eq. 2.7, we easily see that for every $p \in [1, \infty]$, $\mathbb{S}_p$ is an SDD-type class, we can apply the Compactness and Isolation principles. Therefore, localization sets are always compact, and if disjoint parts occure, we exactly know the number of eigenvalues in each of them.

To illustrate these localization sets, we have plotted Figs. 1 and 2. For the matrix M_2 given in the Example 13, the set $\Theta^{\mathbb{S}_1}(M_2)$ is plotted in Fig. 1a, and the set $\Theta^{\mathbb{S}_2}(M_2)$ in Fig. 1b. For the matrix M_3 of the Example 16, the set $\Theta^{\mathbb{S}_1}(M_3)$ is plotted in Fig. 2a, and the set $\Theta^{\mathbb{S}_\infty}(M_3)$ in Fig. 2b.

We continue by establishing the Inertia principle. Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, let n_+ (n_-) denote the number of eigenvalues of A which have positive (negative) real part. If we denote dimension of the kernel of A by n_0, then the triple $in(A) := (n_+, n_0, n_-)$, is called **inertia** of the general complex matrix A. This matrix property is widely studied due to its connection with stability of dynamical systems, [17]. For the real SDD and nonsingular H-matrices, it is known that their inertia is completely described by the position of its diagonal entries. In [41] a special case when $n_- = n$ was studied. Here, we show how the properties of inertia can easily be established through the use of Geršgorin type localization sets.

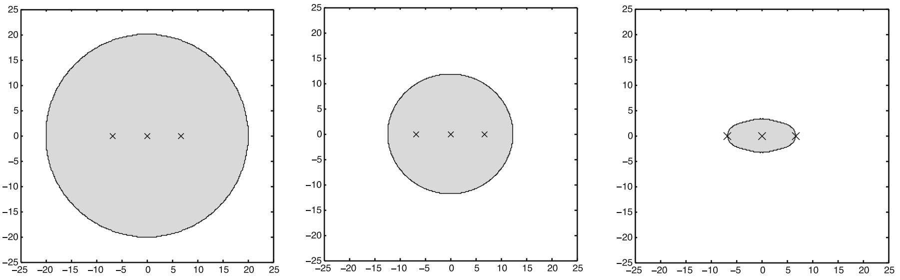
Theorem 11 (Inertia Principle - real case) *Given a nonsingular SDD-type class $\mathbb{K}$, an arbitrary real matrix $A \in \mathbb{R}^{n,n}$, $n \geq 2$, and the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$, let d_+ and d_- denote the number of positive and negative diagonal entries of A, respectively. Then, $A \in \mathbb{K}$ if and only if $\Theta^{\mathbb{K}}(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, implying that $in(A) = (d_+, 0, d_-)$.*

Proof Obviously, due to definition of the localization set, $A \in \mathbb{K}$ is equivalent to $0 \notin \Theta^{\mathbb{K}}(A)$. So, it remains to prove that if $A \in \mathbb{K}$ then for every $t \in \mathbb{R}$, $\mathrm{i}t \notin \Theta^{\mathbb{K}}(A)$.



a: $\mathbb{K} = \mathbb{S}_1$ b: $\mathbb{K} = \mathbb{S}_2$ c: $\mathbb{K} = \mathbb{H}$

Fig. 1 Geršgorin-type sets $\Theta^{\mathbb{K}}(M_2)$ of the matrix M_2 of Example 13 for three different SDD-type classes $\mathbb{K}$. The exact eigenvalues of M_2 are marked by $\times$



a: $\mathbb{K} = \mathbb{S}_1$ b: $\mathbb{K} = \mathbb{S}_\infty$ c: $\mathbb{K} = \mathbb{H}$

Fig. 2 Geršgorin-type sets $\Theta^{\mathbb{K}}(M_3)$ of the matrix B of Example 16 for three different SDD-type classes $\mathbb{K}$. The exact eigenvalues of M_3 are marked by $\times$

But, since $\langle \mathtt{i}tI - D\rangle \geq \langle D\rangle$, for every diagonal matrix D and every real number t, $\langle \mathtt{i}tI - A\rangle \geq \langle A\rangle$. Now, since $\mathbb{K}$ is an SDD-type class, $\langle \mathtt{i}tI - A\rangle \in \mathbb{K}$ for all $t \in \mathbb{R}$ and, therefore, $\Theta^{\mathbb{K}}(A)$ doesn't intersect imaginary axis. The remaining part of the theorem follows from the Compactness and Isolation principles. □

It is interesting to note that Theorem 11 remains valid if the matrix has complex off-diagonal entries. But, if a diagonal entry is a complex number, in general, it is not true. So, in this case we have a different behaviour. Given a complex number $z \in \mathbb{C}$, let $\Re(z)$ and $\Im(z)$ denote its real and imaginary part, respectively. Then, for a given complex matrix $A \in \mathbb{C}^{n,n}$, $n \geq 2$, we define the matrix $\mathrm{Red}A = [m_{ij}] \in \mathbb{R}^{n,n}$ by

$$m_{ij} := \begin{cases} \Re(a_{ii}), & i = j, \\ |a_{ij}|, & \text{otherwise.} \end{cases} \tag{3.9}$$

Theorem 12 (Inertia Principle - complex case) *Given a nonsingular SDD-type class $\mathbb{K}$, an arbitrary complex matrix $A \in \mathbb{C}^{n,n}$, $n \geq 2$, and the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$, let let d_+ and d_- denote the number of diagonal entries of A which have positive real part and negative real part, respectively. Then,* $\mathrm{Red}(A) \in \mathbb{K}$ *implies $\Theta^{\mathbb{K}}(A) \cap \mathtt{i}\mathbb{R} = \emptyset$, and, consequently, $in(A) = (d_+, 0, d_-)$.*

Proof As before, since $\langle \mathtt{i}tI - A\rangle \geq \langle \mathrm{Red}(A)\rangle$, for all $t \in \mathbb{R}$, the proof follows from the fact that $\mathbb{K}$ is an SDD-type class combined with Isolation and Compactness principles. □

The following example shows that, in the complex case, $\mathrm{Red}(A) \in \mathbb{K}$ is not a necessary condition, in general.

Example 13 Consider $\mathbb{H}$, the class of nonsingular H-matrices, and the matrix

$$M_2 = \begin{bmatrix} -2.5 + 4\mathtt{i} & 2 & 1 \\ 1 & -2.5 & 1 \\ 1 & 2 & -2.5 - 4\mathtt{i} \end{bmatrix}.$$

Then, it can be easily checked that $\langle \mathrm{Red}(M_2)\rangle^{-1}$ is not nonnegative, implying that $\mathrm{Red}(M_2) \notin \mathbb{H}$. But, the minimal Geršgorin set for this matrix, $\Theta^{\mathbb{H}}(M_2)$, does not intersect imaginary axis, see Fig. 1c. Therefore, the converse of the last theorem cannot hold in general.

An immediate corollary of the Inertia principle is the famous result that every real nonsingular H-matrix is Hurwitz stable if and only if all its diagonal entries are negative. But, in practice, one often needs conditions for stability that are easier to check, see [27]. Namely, to check if the given matrix A is an H-matrix, typically one should either:

a) compute an inverse of $\langle A\rangle$ and check its nonnegativity,
b) apply some of special iterative scaling-based algorithms that may produce result, see [1, 28],
c) compute the left most eigenvalue of $\langle A\rangle$ and check if its negative.

Of course, item a) is to directly use the definition of H-matrices, but this is the most computationally expensive and numerically unstable way to it. On the other hand, algorithms developed in [1, 28] are numerically stable, but, in practice, they can fail to give an answer if the the matrix $\langle A\rangle$ is close to be singular.

While approaches a) and b) where already used in the literature, up to authors knowledge, c) was not. So, we present it as a **novel approach to test if a given matrix is nonsingular H-matrix**. It is essentially based on the characterization of Minimal Geršgorin set given in [50, 51], and Theorem 8. Namely, according to Theorem 8, A is an H-matrix if and only if $0 \notin \Theta^{\mathbb{H}}(A) = \Gamma^{\mathfrak{R}}(A)$. But, according to [50, Proposition 4.3], this is equivalent to the fact that $\nu_A(0) < 0$, where

$$\nu_A(z) := \inf_{x>0} \max_{i\in N} \frac{(-\langle zI - A\rangle x)_i}{x_i}, \quad z \in \mathbb{C}. \tag{3.10}$$

But, following the reasoning in [51], it is easily seen that $\nu_A(0)$ is exactly the left-most eigenvalue of the matrix $\langle A\rangle$, and that it can be computed as a Perron root of the shifted *essentially-nonegative* matrix $-\langle A\rangle$. Therefore, not only that this approach overcomes the drawbacks of a) and b), but also it provides the robustness of the answer as well. Namely, closer computed left-most eigenvalue of $\langle A\rangle$ is to zero, less robust is the answer, i.e., under very small small unstructured perturbations of the matrix A, the obtained answer could change.

So, to summarize, using the Inertia principle, all SDD-type classes can be used to obtain practical necessary conditions for Hurwitz stability in a more general way then before. Moreover, using the information obtained from Geršgorin-type sets, possible nearness of eigenvalues to the imaginary axis, which is an important information that concerns robustness of the stability results, can be established, too. In practice, when one needs necessary conditions to be closed formulas, one may use some of the known subclasses of $\mathbb{H}$, while the most precise result of this kind is obtained with class $\mathbb{H}$ performing the introduced test c).

Another interesting observation is that using the last theorem we can obtain the lines in the complex plane that can separate the spectra by separating their Geršgorin-type sets. This idea is especially meaningful in the case of minimal Geršgorin set which, in general, cannot be plotted easily. Namely, instead of plotting the whole set, see [51], in order to inspect the number of eigenvalues of a given matrix on each side of a certain line, we can simply use the following theorem and avoid unnecessary computation. But, this theorem can also be useful as a cheep test for eigenvalue separation when some other SDD-type classes (SDD, S-SDD, Brualdi SDD, etc.) are used.

Corollary 14 (Eigenvalue separation lines) *Given a nonsingular SDD-type class* $\mathbb{K}$*, an arbitrary complex matrix* $A \in \mathbb{C}^{n,n}$*,* $n \geq 2$*, and arbitrary complex number* $\alpha \in \mathbb{C}$*, if* $\mathrm{Red}(|\alpha| I - e^{-\mathrm{i}\arg(\alpha)} A) \in \mathbb{K}$*, then the corresponding Geršgorin-type set* $\Theta^{\mathbb{K}}(A)$ *does not intersect a line in complex plane* $\left\{\alpha + t e^{\mathrm{i}(\arg(\alpha)+\frac{\pi}{2})} \;:\; t \in \mathbb{R}\right\}$. *Moreover, the number of eigenvalues and diagonal entries of the matrix* A *on each side of the line coincide.*

Proof Simply observe that for a $z \in \mathbb{C}$, $|z - \alpha - t e^{\mathrm{i}(\arg(\alpha)+\frac{\pi}{2})}| = |z e^{-\mathrm{i}\arg(\alpha)} - |\alpha| - \mathrm{i}t|$, and the proof immediately follows. $\square$

To illustrate this, consider again matrix M_2 from Example 13. As one can see on Fig. 1c, lines $\{-2\mathrm{i} + t \;:\; t \in \mathbb{R}\}$ and $\{2\mathrm{i} + t \;:\; t \in \mathbb{R}\}$ do not intersect minimal Geršgorin set of A. Using the previous corollary, this is a consequence of the fact that the following two matrices:

$$\mathrm{Red}(2I - \mathrm{i}M_2) = \begin{bmatrix} 6 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}, \text{ and } \mathrm{Red}(2I + \mathrm{i}M_2) = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 6 \end{bmatrix},$$

are both H-matrices. Furthermore, from these two matrices we can also conclude that each part of the complex plane $\Im(z) < -2$, $-2 < \Im(z) < 2$ and $2 < \Im(z)$ contains exactly one eigenvalue of the matrix A.

As we have seen, the Inertia principle shows how Geršgorin-type sets can be used as a tool for the stability of continuous time dynamical systems through its connection with spectral abscissa. But, on the other hand, stability of discrete time dynamical systems, as well as convergence of fixed point iteration methods, is established through the use of spectral radius (see [46]). Therefore, to conclude this paper, we show how the estimates for spectral radius of a given matrix are obtained using the concept of Geršgorin-type sets.

Given a nonsingular SDD-type class $\mathbb{K}$, and an arbitrary matrix $A \in \mathbb{C}^{n,n}$, we define the value

$$\rho^{\mathbb{K}}(A) := \max\left\{|z| \;:\; zI - A \notin \mathbb{K},\; z \in \mathbb{C}\right\} \tag{3.11}$$

and call it **SDD-type bound** of spectral radius of a matrix A.

That this value is properly defined, i.e., the maximum exists, follows from the Compactness Principle, while the name is justified due to the Equivalence principle. Namely, given an arbitrary matrix $A \in \mathbb{C}^{n,n}$, $\rho(A) \leq \rho^{\mathbb{K}}(A)$. In addition, it is easy to see that all such bounds of a spectral radius are limited by the diagonal entries of a matrix. More precisely, for every $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and every nonsingular SDD-type class $\mathbb{K}$, $\max_{i\in N} |a_{ii}| \le \rho^{\mathbb{K}}(A)$.

Taking a class $\mathbb{K}$ to be class of SDD matrices, we obtain a very well known bound $\rho(A) \le \rho^{\mathbb{K}}(A) = ||A||_\infty$, for all $A \in \mathbb{C}^{n,n}$. Knowing that the monotonicity principle holds, using any nonsingular SDD-type superclass of SDD class, we obtain a better bound. Of course, the best SDD-type bound would be $\rho^{\mathbb{H}}(A)$ which improves very well know estimate $\rho(A) \le \rho(|A|)$. More precisely, according to [50, Proposition 4.3], for every $A \in \mathbb{C}^{n,n}$

$$\rho^{\mathbb{H}}(A) := \max\{|z| \ : \ \nu_A(z) \ge 0, \ z \in \mathbb{C}\}. \tag{3.12}$$

But, from Eq. 3.10 we have that

$$\nu_A(z) = \inf_{x>0} \max_{i\in N} \frac{([-|zI - D_A| + |F_A|]x)_i}{x_i},$$

implying that

$$\nu_A(z) \le \inf_{x>0} \max_{i\in N} \frac{([-|z|I + |D_A| + |F_A|]x)_i}{x_i} = \rho(|A|) - |z|.$$

Therefore, we have obtained that

$$\rho(A) \le \rho^{\mathbb{H}}(A) \le \rho(|A|)$$

holds or every matrix $A \in \mathbb{C}^{n,n}$. As Example 16 shows the bound by $\rho^{\mathbb{H}}(A)$ can be sharp and strictly better than $\rho(|A|)$.

So, a natural motivation is to give formulas for other SDD-type classes. Here, as an illustration of the technique, we give formula for the Ostrowski SDD matrices of the first type (also known as α1-matrices, see [8]). For some other well known SDD-type classes formulas for Eq. 3.11 could be obtained in the similar way.

Let $\mathbb{O}$ be a class of Ostrowski SDD matrices of the first type, i.e., $A = [a_{ij}] \in \mathbb{O}$ if and only if there exists a parameter $\alpha \in [0, 1]$, such that

$$|a_{ii}| > \alpha r_i(A) + (1-\alpha)c_i(A) \ \text{ for all } \ i \in N, \tag{3.13}$$

where $c_i(A) := r_i(A^T)$ denotes i-th deleted column sum of the matrix A.

The nonsingularity of all matrices in this class is proved in [39]. So, since by inspection we see that this is an SDD-type class, as a corollary of Theorem 2 we have that $\mathbb{O} \subseteq \mathbb{H}$, and all the principles of this paper apply. Here, we will just illustrate how an upper bound for a spectral radius is obtained. Given an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, the goal is to find a maximal modulus of a complex number $z \in \mathbb{C}$ (in the limit case) such that for all $\alpha \in [0, 1]$, there exists $i \in N$, such that $|z - a_{ii}| \le \alpha r_i(A) + (1-\alpha)c_i(A)$. But, obviously, for a fixed α, this is achieved with $z \in \mathbb{C}$, such that $|z - a_{kk}| = |z| - |a_{kk}|$, for some $k \in N$, and $|z| \ge |a_{ii}| + \alpha r_i(A) + (1-\alpha)c_i(A)$, for all $i \in N$. Therefore, searched supremum can be expressed as:

$$\rho^{\mathbb{O}}(A) = \min_{\alpha\in[0,1]} \max_{i\in N} \{|a_{ii}| + \alpha r_i(A) + (1-\alpha)c_i(A)\}. \tag{3.14}$$

So, using different values of the parameter α one can establish different bounds for spectral radius of A, where the best bound is, of course given by (3.14). If we want to find the formula for $\rho^{\mathbb{O}}(A)$, or at least its tight upper bound, that doesn't depend of the parameter α, we can use the results of [12] and [4].

Theorem 15 [12] *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, with $n \geq 2$, is an α_1-matrix if and only if the following two conditions hold:*

(i) $|a_{ii}| > \min\{r_i(A), c_i(A)\}$, *for all* $i \in N$,

(ii) $\dfrac{|a_{ii}| - c_i(A)}{r_i(A) - c_i(A)} > \dfrac{c_j(A) - |a_{jj}|}{c_j(A) - r_j(A)}$, *for all* $i \in \mathcal{R}$, *and all* $j \in \mathcal{C}$,

where $\mathcal{R} := \{i \in N : \ r_i(A) > c_i(A)\}$, *and* $\mathcal{C} := \{i \in N : \ c_i(A) > r_i(A)\}$.

Now, rewriting item (ii) as

$$[|a_{ii}| - c_i(A)][c_j(A) - r_j(A)] + [|a_{jj}| - c_j(A)][r_i(A) - c_i(A)] > 0,$$

for all $i \in \mathcal{R}$ and all $j \in \mathcal{C}$, we can, as before, conclude that for Eq. 3.14 holds:

$$\rho^{\mathbb{O}}(A) \leq \hat{\rho}^{\mathbb{O}}(A) := \max\{f(A), g(A)\}, \tag{3.15}$$

where

$$f(A) := \max_{k \in N}[|a_{kk}| + \min\{r_k(A), c_k(A)\}]$$

and

$$g(A) := \max_{i \in \mathcal{R}} \max_{j \in \mathcal{C}} \frac{[|a_{ii}| + c_i(A)][c_j(A) - r_j(A)] + [|a_{jj}| + c_j(A)][r_i(A) - c_i(A)]}{c_j(A) - r_j(A) + r_i(A) - c_i(A)}.$$

Considering the matrix M_2 from Example 13, one can see that $\rho(M_2) = 4.2752$, while the obtained bound is $\hat{\rho}^{\mathbb{O}}(M_2) = 6.7170$, which only slightly improved the one by SDD class $||M_2||_\infty = 7.4721$. As said before, for this example, using the SDD-type classes we cannot get a bound that is less then maximal modulus of a diagonal entry, i.e., $2\sqrt{5} = 4.4721$. More precisely, the best bound through use of SDD-type classes is obtained by class of nonsingular H-matrices, i.e., Minimal Geršgorin set, which here gives a bound $\rho^{\mathbb{H}}(M_2) = 5.1442$. But, in general, these bounds can be significantly sharper.

Example 16 Let

$$M_3 = \begin{bmatrix} 5 & 1 & 0 \\ 10 & 0 & 10 \\ 0 & 1 & -5 \end{bmatrix}.$$

Then, the best SDD-type bound is sharp, since $\rho(M_3) = \rho^{\mathbb{H}}(M_3) = 6.7082$, while $\rho(|M_3|) = 7.6235$ is not. The bounds obtain by other SDD-type classes we considered in this paper are $\hat{\rho}^{\mathbb{O}}(M_3) = 10.6667$, $\rho^{\mathbb{S}_1}(M_3) = ||M_3||_\infty = 20$, and $\rho^{\mathbb{S}_\infty}(M_3) = 12.391$. Note that in this example, set $\Theta^{\mathbb{S}_p}(M_3)$ is the smallest for $p = \infty$, thus, the class $\mathbb{S}_\infty$ produces the best bound of all $\mathbb{S}_p$ classes. For $\mathbb{K} \in \{\mathbb{S}_1, \mathbb{S}_\infty, \mathbb{H}\}$, the localization sets that produce these bounds are plotted in Fig. 2.

As a final remark of this paper, we present another principle concerning DD-type bounds of spectral radius. Namely, it is well known that a matrix A is SDD if and only if $||D^{-1}F||_\infty < 1$, where $A = D - F$ is a splitting of a matrix to its diagonal part

D and off-diagonal part F. This nice property implies that Jacobi iterative method for every SDD system converges, and it was also extensively used in the convergence theory of other relaxation iterative methods (see as an example [10, 11]). But, does this hold for some other SDD-type classes, too? The following theorem clarifies this.

Theorem 17 (Jacoby Principle) *Let $\mathbb{K}$ be a nonsingular SDD-type class that is invariant under diagonal scaling of rows, i.e., $A \in \mathbb{K}$ implies that for every nonsingular diagonal matrix D, $DA \in \mathbb{K}$.*

Then, for every matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $A \in \mathbb{K}$ if and only if $\rho^{\mathbb{K}}(D^{-1}F) < 1$, where $A = D - F$ and $D := \mathrm{diag}(a_{11}, a_{22}, ..., a_{nn})$.

Proof First, let $A \in \mathbb{K}$, then $D^{-1}A = I - D^{-1}F \in \mathbb{K}$, and since $\mathbb{K}$ is an open class, there exists a small $\varepsilon > 0$ such that $(1-\varepsilon)I - D^{-1}F \in \mathbb{K}$. Therefore, $\rho^{\mathbb{K}}(D^{-1}F) < 1$. On the other hand, if $\rho^{\mathbb{K}}(D^{-1}F) < 1$, then there exists $z \in \mathbb{C}$, such that $|z| < 1$ and $zI - D^{-1}F \in \mathbb{K}$. But, then, $\langle zI - D^{-1}F\rangle < \langle D^{-1}A\rangle$, implying that $A \in \mathbb{K}$. □

Since the classes $\mathbb{S}_p$, for $p \in [0, \infty]$, are invariant under diagonal scaling of rows, Jacobi principle holds for them. On the other hand, class $\mathbb{O}$ is not invariant under scaling of rows, and, consequently this principle does not hold.

Acknowledgments The author would like specially to thank to Ljiljana Cvetković and Richard S. Varga for their experience that much helped in developing the results of this paper, and more importantly to their immense influence to his research in general.

The author is partially supported by Ministry of Science, research grant 174019, Provincial Secretariat for Science of Vojvodina, research grants 1850 and 2492, and the Einstein Fellowship Foundation of Berlin Mathematical School for 2013/14.

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