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On linear quadratic optimal control of discrete-time complex-valued linear systems

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Summary

We study in this paper the linear quadratic optimal control (linear quadratic regulation, LQR for short) for discrete-time complex-valued linear systems, which have several potential applications in control theory. Firstly, an iterative algorithm was proposed to solve the discrete-time bimatrix Riccati equation associated with the LQR problem. It is shown that the proposed algorithm converges to the unique positive definite solution (bimatrix) to the bimatrix Riccati equation with appropriate initial conditions. With the help of this iterative algorithm, the LQR problem for the antilinear system, which is a special case of complex-valued linear system, was carefully examined and three different Riccati equations–based approaches were provided, namely, bimatrix Riccati equation, anti-Riccati equation, and normal Riccati equation. The established approach is then used to solve the LQR problem for a discrete-time time-delay system with one-step state delay, and a numerical example was used to illustrate the effectiveness of the proposed methods.

KEYWORDS

bimatrix, complex-valued linear systems, linear optimal control, Riccati equations, time-delay systems

1 | INTRODUCTION

Complex-valued linear systems refer to linear systems whose right-hand side is dependent on both the state and its conjugate.[1] We study complex-valued linear systems because they have several potential applications in control theory, for example, describing linear dynamical quantum systems,[2] second-order dynamical systems,[3] symmetric linear systems,[3] and time-delay systems (see Section 4 in this paper). Recently, we have studied several analysis and design problems for complex-valued linear systems, including state response, controllability, observability, stability, pole assignment, stabilization, linear quadratic regulation (LQR) and observer design.[1] We have shown that, with the help of the so-called bimatrix, results obtained for complex-valued linear systems are quite analogous to those for normal linear systems.[1] Moreover, we have shown that the obtained results include those for normal linear systems[4] and antilinear systems,[5,6] which are particular cases of complex-valued linear systems, as special cases.[1]

The LQR problem is a fundamental problem in both linear systems theory and optimal control theory, and has been extensively investigated in the literature.[7,8] For the infinite-time LQR problem, it has been well known that the solution is completely characterized by the associated algebraic Riccati equation.[7-9] The LQR problem has been extended to several different situations. For example, LQR control with time delay was investigated in the works of Liang and Zhang[10] and Wu and Shu[11]; the LQR problem for stochastic systems was studied in the works of Liu et al[12] and Wu and Zhuang[13]; and LQR for stochastic time-delay systems was solved in the work of Li et al.[14] In the work of Wu et al,[15] the LQR problem was solved for the so-called antilinear system (which is a special case of the complex-valued linear systems), and a so-called anti-Riccati equation–based solution was established.

With the help of the concept of bimatrix, we have recently solved the LQR problem for complex-valued linear systems.[1] It was shown that the existence of an optimal solution is equivalent to the stabilizability of the complex-valued linear systems and is also equivalent to the existence of positive definite bimatrix to some bimatrix Riccati equation.[1] In this paper, based on our early work, we continue to study the LQR problem for discrete-time complex-valued linear systems. We first establish an iterative algorithm for solving the discrete-time bimatrix Riccati equation. The convergence of the algorithm is proven. This iterative algorithm is not only useful for computing the solution (a bimatrix) to the bimatrix Riccati equation but is also helpful in establishing theoretical results for the so-called anti-Riccati equation associated with the LQR problem for antilinear systems. Indeed, with such an iterative algorithm, we have shown that, under the stabilizability assumption, the existence of a solution to the LQR problem for antilinear systems is equivalent to the existence of a positive definite solution to the anti-Riccati equation, which closes the gap in the work of Wu et al,[15] where the existence of a positive definite solution to the ani-Riccati equation was not guaranteed. We will also establish another normal Riccati equation–based solution to the LQR problem for antilinear systems. The relationships among the bimatrix Riccati equation, anti-Riccati equation, and normal Riccati equation are revealed. At the same time, we show that the anti-Riccati equation can be equivalently transformed into a nonlinear matrix equation that has been carefully studied in our early work.[16,17] Finally, by expressing a discrete-time linear time-delay system as a complex-valued system model, the LQR problem for such a system is solved by using bimatrix Riccati equations. A numerical example was worked out to illustrate the effectiveness of the proposed approach.

The remainder of this paper is organized as follows. In Section 2, after reviewing briefly the complex-valued system model and the bimatrix Riccati equation–based solution to the LQR problem, we establish an iterative algorithm and prove its convergence. Then, in Section 3, the LQR problem for the antilinear system will be carefully studied. The LQR theory for complex-valued linear systems was then used in Section 4 to study the LQR problem for discrete-time time-delay systems, and a numerical example will also be provided. This paper concludes in Section 5.

Notation. For a matrix $A \in \mathbf{C}^{n\times m}$, we use $A^{\#}$, A^{T}, A^{H}, rank (A), $\|A\|$, Re (A), and Im (A) to denote respectively its conjugate, transpose, conjugate transpose, rank, norm, real part, and imaginary part. Thus, $A^{-\#}$ denotes $(A^{\#})^{-1}$ or $(A^{-1})^{\#}$. Denote by j the unitary imaginary number. For a matrix pair $(A_1, A_2) \in (\mathbf{C}^{n\times m}, \mathbf{C}^{n\times m})$, the bimatrix $\{A_1, A_2\}$ is defined in such a manner that $\{A_1, A_2\}x = A_1x + A_2^{\#}x^{\#}$. Further definitions and properties about bimatrix are collected in the Appendix.

2 | OPTIMAL CONTROL OF COMPLEX-VALUED LINEAR SYSTEMS

2.1 | A brief introduction to complex-valued linear systems

We continue to study in this paper the following complex-valued linear system[1,3]:

$$x(k+1) = \{A_1, A_2\}x(k) + \{B_1, B_2\}u(k), \tag{1}$$

where $A_i \in \mathbf{C}^{n\times n}$ and $B_i \in \mathbf{C}^{n\times m}, i = 1, 2$, are known coefficients; $x(k)$ is the state; and $u(k)$ is the control. The initial condition is set to be $x(0) = x_0 \in \mathbf{C}^n$. Clearly, system (1) becomes the normal linear system

$$x(k+1) = A_1x(k) + B_1u(k), \tag{2}$$

if A_2 and B_2 are null, and becomes the so-called antilinear system

$$x(k+1) = A_2^{\#}x^{\#}(k) + B_2^{\#}u^{\#}(k), \tag{3}$$

if A_1 and B_1 are zeros. The antilinear system (3) was firstly studied in the works of Wu et al.[5,6] We have shown recently in our other works[1,3] that the complex-valued linear system has several potential applications in control, for example, for control of linear dynamical quantum systems[2] and second-order dynamical systems.[3]
In this paper, based on our early work,[1] we continue to study the linear quadratic optimal control problem for system (1).

To this end, we introduce some based concepts for this system.

Definition 1 (See our other work[1]). The complex-valued linear system (1) is said to be stabilizable if there exists a so-called full-state feedback

$$u(k) = \{K_1, K_2\}\, x(k) = K_1 x(k) + K_2^{\#} x^{\#}(k) \tag{4}$$

such that the following closed-loop system is asymptotically stable:

$$x(k+1) = (\{A_1, A_2\} + \{B_1, B_2\}\,\{K_1, K_2\})\, x(k). \tag{5}$$

The following result was proven in our other work.[1]

Lemma 1. *The complex-valued linear system (1) is stabilizable if and only if*

$$\mathit{rank} \quad \begin{bmatrix} \lambda I_n - A_1 & -A_2^{\#} & B_1 & B_2^{\#} \\ -A_2 & \lambda I_n - A_1^{\#} & B_2 & B_1^{\#} \end{bmatrix} = 2n,\ \forall \lambda \in \{s : |s| \geq 1\}.$$

The following simple test for the stabilizability of the antilinear system (3) was also recalled from our other work.[1]

Corollary 1. *The antilinear system (3) is stabilizable if and only if*

$$\mathrm{rank}\, \left[\lambda I_n \quad - A_2 A_2^{\#}\ \ B_2\ \ A_2 B_2^{\#}\right] = n,\ \forall \lambda \in \{s : |s| \geq 1\}, \tag{6}$$

namely, the normal discrete-time linear system $\left(A_2 A_2^{\#}, \left[B_2, A_2 B_2^{\#}\right]\right)$ *is stabilizable.*

It follows that, for stabilization of the complex-valued linear system (1), the full-state feedback (4) is generally necessary. However, for the discrete-time antilinear system (3), the well-used normal linear feedback

$$u(k) = K_1 x(k), \tag{7}$$

is enough for stabilization under condition (6).[1]

2.2 | Problem formulation and solution

We study the LQR problem for the complex-valued linear system (1). Consider the real-valued quadratic index function

$$J(u) = \sum_{k=0}^{\infty} \left(x^{\mathrm{H}}(k) Q x(k) + u^{\mathrm{H}}(k) R u(k)\right), \tag{8}$$

where $Q \in \mathbf{C}^{n\times n}$ and $R \in \mathbf{C}^{m\times m}$ are given positive definite weighting matrices (Q can be semipositive definite; however, we assume $Q > 0$ for simplicity). The LQR problem refers to as finding an optimal controller u^* for system (1) such that $J(u)$ is minimized, denoted by $J_{\min}(u^*)$. The LQR problem is said to be solvable if $J_{\min}(u^*) < \infty$.[1,7]
The following result was proven in our other work,[1] regarding the existence of a solution to the LQR problem.

Lemma 2. *The following statements are equivalent:*

1. *The LQR problem associated with system (1) has a solution.*
2. *The complex-valued linear system (1) is stabilizable.*
3. *There is a unique bimatrix* $\{P_1, P_2\} > 0$ *to the following bimatrix Riccati equation*

$$\begin{aligned} -\{Q, 0\} =& \{A_1, A_2\}^{\mathrm{H}}\{P_1, P_2\}\{A_1, A_2\} - \{P_1, P_2\} \\ & - \{A_1, A_2\}^{\mathrm{H}}\{P_1, P_2\}\{B_1, B_2\}\{S_1, S_2\}^{-1}\{B_1, B_2\}^{\mathrm{H}}\{P_1, P_2\}\{A_1, A_2\}, \end{aligned} \tag{9}$$

where $\{S_1, S_2\} = \{R, 0\} + \{B_1, B_2\}^{\mathrm{H}}\{P_1, P_2\}\{B_1, B_2\}$.

Under one of the above conditions, the optimal control is the full-state feedback

$$u^*(k) = \left\{K_1^*, K_2^*\right\} x(k), \tag{10}$$

where $\{K_1^*, K_2^*\}$ is the optimal feedback gain bimatrix determined by

$$\{K_1^*, K_2^*\} = -\{S_1, S_2\}^{-1}\{B_1, B_2\}^{\mathrm{H}}\{P_1, P_2\}\{A_1, A_2\}, \tag{11}$$

the closed-loop system is asymptotically stable, and the minimal value of $J(u)$ is given by

$$J_{\min}(u^*) = Re\left(x_0^{\mathrm{H}}\{P_1, P_2\}x_0\right). \tag{12}$$

The above result is quite neat in the sense that the bimatrix Riccati equation takes an analogous form as the usual Riccati matrix equation.[9]

2.3 | Iterative solution to the bimatrix Riccati equation

In this section, we provide an iterative method for solving the bimatrix Riccati equation (9). This method is not only useful for computing solutions to (9) but is also helpful in proving theoretical results in the subsequent sections.

Motivated by the existing work for normal discrete-time Riccati equations,[18] we construct the following iteration associated with the bimatrix Riccati equation (9):

$$\begin{aligned}&\{P_1(k+1), P_2(k+1)\} = \{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\{P_1(k), P_2(k)\}\{A_1, A_2\}\\&- \{A_1, A_2\}^{\mathrm{H}}\{P_1(k), P_2(k)\}\{B_1, B_2\}\{S_1(k), S_2(k)\}^{-1}\{B_1, B_2\}^{\mathrm{H}}\{P_1(k), P_2(k)\}\{A_1, A_2\},\end{aligned} \tag{13}$$

where $\{P_1(0), P_2(0)\} = \{Q, 0\}$ and

$$\{S_1(k), S_2(k)\} = \{R, 0\} + \{B_1, B_2\}^{\mathrm{H}}\{P_1(k), P_2(k)\}\{B_1, B_2\}.$$

For notation simplicity, we also denote

$$\{R_1, R_2\} = \{B_1, B_2\}\{R, 0\}^{-1}\{B_1, B_2\}^{\mathrm{H}}. \tag{14}$$

Theorem 1. *Assume that the complex-valued linear system (1) is stabilizable and $\{P_1, P_2\}$ is the unique positive definite solution to (9). Then, for any $k \geq 0$,*

$$\{Q, 0\} \leq \{P_1(k), P_2(k)\} \leq \{P_1(k+1), P_2(k+1)\} \leq \{P_1, P_2\}. \tag{15}$$

Consequently, the limit of $\{P_1(k), P_2(k)\}$ as k approaches infinity exists and

$$\{P_1, P_2\} = \lim_{k\to\infty}\{P_1(k), P_2(k)\}. \tag{16}$$

Proof. We first show

$$\{Q, 0\} \leq \{P_1(k), P_2(k)\} \leq \{P_1(k+1), P_2(k+1)\}, \tag{17}$$

by mathematical induction. We have from Lemma 3 that

$$\begin{aligned}\{P_1(1), P_2(1)\} =&\{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\{Q, 0\}\{A_1, A_2\}\\&- \{A_1, A_2\}^{\mathrm{H}}\{Q, 0\}\{B_1, B_2\}\{S_1(0), S_2(0)\}^{-1}\{B_1, B_2\}^{\mathrm{H}}\{Q, 0\}\{A_1, A_2\}\\=&\{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\left(\{Q, 0\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\}\\\geq&\{Q, 0\} = \{P_1(0), P_2(0)\},\end{aligned} \tag{18}$$

which means that (17) is true with $k = 0$. We now assume that (17) is true with $k = i$, namely, $\{Q, 0\} \leq \{P_1(i), P_2(i)\} \leq \{P_1(i+1), P_2(i+1)\}$, or equivalently, $\{P_1(i), P_2(i)\}^{-1} \geq \{P_1(i+1), P_2(i+1)\}^{-1}$. Therefore,

$$\left(\{P_1(i+1), P_2(i+1)\}^{-1} + \{R_1, R_2\}\right)^{-1} \geq \left(\{P_1(i), P_2(i)\}^{-1} + \{R_1, R_2\}\right)^{-1}.$$

On the other hand, by Lemma 3, we can write the iteration (13) as

$$\{P_1(j+1), P_2(j+1)\} = \{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1(j), P_2(j)\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\}, \tag{19}$$

where $j = i$ and $i + 1$. Then,

$$\begin{aligned}
&\{P_1(i+2), P_2(i+2)\} - \{P_1(i+1), P_2(i+1)\} \\
&= \{A_1, A_2\}^{\mathrm{H}}\left(\left(\{P_1(i+1), P_2(i+1)\}^{-1} + \{R_1, R_2\}\right)^{-1} - \left(\{P_1(i), P_2(i)\}^{-1} + \{R_1, R_2\}\right)^{-1}\right)\{A_1, A_2\} \\
&\geq 0,
\end{aligned} \tag{20}$$

which implies that (17) is satisfied with $k = i + 1$.

We next show

$$\{P_1(k), P_2(k)\} \leq \{P_1, P_2\}, \tag{21}$$

also by mathematical introduction. It follows from

$$\begin{aligned}
\{P_1, P_2\} &= \{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1, P_2\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\} \\
&\geq \{Q, 0\} = \{P_1(0), P_2(0)\}
\end{aligned} \tag{22}$$

that (21) is satisfied with $k = 0$. Assume that (21) is true with $k = i$, namely, $\{P_1(i), P_2(i)\} \leq \{P_1, P_2\}$. Then, similarly to (20), we have from (19) and (22) that

$$\begin{aligned}
\{P_1, P_2\} - \{P_1(i+1), P_2(i+1)\} =& \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1, P_2\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\} \\
&- \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1(i), P_2(i)\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\} \geq 0,
\end{aligned}$$

which shows (21) with $k = i + 1$. Thus, (15) is proven.

Finally, (16) follows from (15) because the bimatrix Riccati equation (9) has only a unique positive definite solution. The proof is finished. □

By (15), we can see that, with Lemma 3, the iteration (13) can also be written as

$$\{P_1(k+1), P_2(k+1)\} = \{Q, 0\} + \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1(k), P_2(k)\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\}, \tag{23}$$

where $\{P_1(0), P_2(0)\} = \{Q, 0\}$ and $\{R_1, R_2\}$ is given by (14).

3 | OPTIMAL CONTROL OF ANTILINEAR SYSTEMS

In this section, we are interested in the antilinear system (3). Because it possesses a special structure, more specific results can be obtained.

3.1 | The anti-Riccati equation–based approach

We first present a so-called anti-Riccati equation–based approach.

Theorem 2. *Consider the antilinear system (3). Then, the following three statements are equivalent:*

1. *The LQR problem associated with system (3) has a solution.*
2. *The system (3) is stabilizable, namely, (6) is satisfied.*
3. *There is a unique positive definite solution $P_A > 0$ to the so-called anti-Riccati equation*

$$-Q = A_2^{\mathrm{H}} P_A^{\#} A_2 - A_2^{\mathrm{H}} P_A^{\#} B_2\left(R + B_2^{\mathrm{H}} P_A^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} P_A^{\#} A_2 - P_A. \tag{24}$$

In this case, the unique positive definite solutions to (9) and (24) are related with

$$\{P_1, P_2\} = \{P_A, 0\}. \tag{25}$$

Moreover, the optimal controller is the normal-state feedback (7) with $K_1 = K_1^*$ defined by

$$K_1^* = -\left(R + B_2^{\mathrm{H}} P_A^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} P_A^{\#} A_2, \tag{26}$$

the closed-loop system is asymptotically stable, and the optimal value of $J(u)$ is

$$J_{\min}(u) = x_0^{\mathrm{H}} P_A x_0. \tag{27}$$

Proof. In view of Lemma 2, we need only to show the equivalence of Item 3 of Lemma 2 and Item 3 of this theorem. Let the bimatrix Riccati equation (9) have a positive definite solution $\{P_1, P_2\}$. The bimatrix $\{R_1, R_2\}$ defined in (14) associated with the antilinear system (3) is given by

$$\begin{aligned}\{R_1, R_2\} &= \{0, B_2\}\{R^{-1}, 0\}\{0, B_2\}^{\mathrm{H}} \\ &= \left\{0, B_2 R^{-1}\right\}\left\{0, B_2^{\mathrm{T}}\right\} \\ &= \left\{B_2^{\#} R^{-\#} B_2^{\mathrm{T}}, 0\right\},\end{aligned}$$

and the iteration (23) for the associated bimatrix Riccati equation (9) can be simplified as

$$\{P_1(k+1), P_2(k+1)\} = \{Q, 0\} + \left\{0, A_2^{\mathrm{T}}\right\}\left(\{P_1(k), P_2(k)\}^{-1} + \left\{B_2^{\#} R^{-\#} B_2^{\mathrm{T}}, 0\right\}\right)^{-1}\{0, A_2\}, \tag{28}$$

where $\{P_1(0), P_2(0)\} = \{Q, 0\}$.

In the following, we will show that

$$P_1(k) > 0,\ P_2(k) = 0,\ \forall k \geq 0. \tag{29}$$

We show this by mathematical induction. Clearly, (29) is satisfied with $k = 0$. Assume that (29) is true with $k = i$. Then, for $k = i + 1$, we have from (28) that

$$\begin{aligned}\{P_1(i+1), P_2(i+1)\} &= \{Q, 0\} + \left\{0, A_2^{\mathrm{T}}\right\}\left(\{P_1(i), 0\}^{-1} + \left\{B_2^{\#} R^{-\#} B_2^{\mathrm{T}}, 0\right\}\right)^{-1}\{0, A_2\} \\ &= \{Q, 0\} + \left\{0, A_2^{\mathrm{T}}\right\}\left\{P_1^{-1}(i) + B_2^{\#} R^{-\#} B_2^{\mathrm{T}}, 0\right\}^{-1}\{0, A_2\} \\ &= \{Q, 0\} + \left\{A_2^{\mathrm{H}}\left(P_1^{-\#}(i) + B_2 R^{-1} B_2^{\mathrm{H}}\right)^{-1} A_2, 0\right\} \\ &= \left\{Q + A_2^{\mathrm{H}}\left(P_1^{-\#}(i) + B_2 R^{-1} B_2^{\mathrm{H}}\right)^{-1} A_2, 0\right\},\end{aligned} \tag{30}$$

which means that $P_2(i+1) = 0$ and $P_1(i+1) > 0$, namely, (29) is true with $k = i + 1$.

As a result, by Theorem 1, we must have

$$\{P_1, P_2\} = \lim_{k\to\infty}\{P_1(k), P_2(k)\} = \left\{\lim_{k\to\infty} P_1(k), 0\right\},$$

namely, $P_1 > 0$ and $P_2 = 0$, which satisfies the bimatrix Riccati equation (9), namely,

$$\begin{aligned}-\{Q, 0\} =& \{0, A_2\}^{\mathrm{H}}\{P_1, 0\}\{0, A_2\} - \{P_1, 0\} \\ &- \{0, A_2\}^{\mathrm{H}}\{P_1, 0\}\{0, B_2\}\{S_1, S_2\}^{-1}\{0, B_2\}^{\mathrm{H}}\{P_1, 0\}\{0, A_2\},\end{aligned} \tag{31}$$

where

$$
\begin{aligned}
\{S_1, S_2\} &= \{R, 0\} + \{0, B_2\}^{\mathrm{H}}\{P_1, 0\}\{0, B_2\} \\
&= \{R, 0\} + \left\{0, B_2^{\mathrm{T}} P_1\right\}\{0, B_2\} \\
&= \left\{R + B_2^{\mathrm{H}} P_1^{\#} B_2, 0\right\}.
\end{aligned} \tag{32}
$$

By the definition of the product of bimatrices, (31) is just

$$
\begin{aligned}
-\{Q, 0\} &= \left\{A_2^{\mathrm{H}} P_1^{\#} A_2, 0\right\} - \{P_1, 0\} - \left\{A_2^{\mathrm{H}} P_1^{\#} B_2, 0\right\}\left\{R + B_2^{\mathrm{H}} P_1^{\#} B_2, 0\right\}^{-1}\left\{B_2^{\mathrm{H}} P_1^{\#} A_2, 0\right\} \\
&= \left\{A_2^{\mathrm{H}} P_1^{\#} A_2, 0\right\} - \{P_1, 0\} - \left\{A_2^{\mathrm{H}} P_1^{\#} B_2 \left(R + B_2^{\mathrm{H}} P_1^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} P_1^{\#} A_2, 0\right\} \\
&= \left\{A_2^{\mathrm{H}} P_1^{\#} A_2 - A_2^{\mathrm{H}} P_1^{\#} B_2 \left(R + B_2^{\mathrm{H}} P_1^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} P_1^{\#} A_2 - P_1, 0\right\}.
\end{aligned}
$$

Hence, the anti-Riccati equation (24) also has a positive definite solution $P_A = P_1$.

On the other hand, if the anti-Riccati equation (24) has a different positive definite solution P_A^*, then it follows from (31) and (32) that $\{P_A^*, 0\}$ is another positive definite solution to the bimatrix Riccati equation (9), which contradicts with Lemma 2. Hence, the positive definite solution to (24) is unique.

Conversely, if the anti-Riccati equation (24) has a positive definite solution P_A, then, as shown above, the bimatrix Riccati equation (9) has a positive definite solution $\{P_A^*, 0\}$, which, by Lemma 2, must be the unique solution. Thus, the equivalence of Item 3 of Lemma 2 and Item 3 of this theorem is proven. Finally, the relationship (25) follows obviously.

The expression (26) and $J_{\min}(u^*)$ have been proven in our other work[1] and we provide the proofs here for completeness. In view of (11), (25), and (32), the optimal gain can be computed as

$$
\begin{aligned}
\left\{K_1^*, K_2^*\right\} &= -\left\{R + B_2^{\mathrm{H}} P_A^{\#} B_2, 0\right\}^{-1}\{0, B_2\}^{\mathrm{H}}\{P_A, 0\}\{0, A_2\} \\
&= -\left\{\left(R + B_2^{\mathrm{H}} P_A^{\#} B_2\right)^{-1}, 0\right\}\left\{B_2^{\mathrm{H}} P_A^{\#} A_2, 0\right\}, \\
&= -\left\{\left(R + B_2^{\mathrm{H}} P_A^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} P_A^{\#} A_2, 0\right\},
\end{aligned}
$$

which is just (26) by noting (10). At last, it follows from (12) and (25) that

$$
\begin{aligned}
J_{\min}(u^*) &= \mathrm{Re}\left(x_0^{\mathrm{H}}\{P_A, 0\} x_0\right) \\
&= \frac{1}{2}\begin{bmatrix} x_0 \\ x_0^{\#} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} P_A & 0 \\ 0 & P_A^{\#} \end{bmatrix} \begin{bmatrix} x_0 \\ x_0^{\#} \end{bmatrix} \\
&= x_0^{\mathrm{H}} P_A x_0.
\end{aligned}
$$

The proof is finished. □

Theorem 2 improves some results in the work of Wu et al[15] and our other work,[1] where the existence of a positive definite solution to (24) was not guaranteed. Moreover, we have relaxed controllability in the work of Wu et al[15] as stabilizability in this paper.

Remark 1. As a by-product of the proof of Theorem 2, we see from (30) that the iteration

$$
P_A(k+1) = Q + A_2^{\mathrm{H}}\left(P_A^{-\#}(k) + B_2 R^{-1} B_2^{\mathrm{H}}\right)^{-1} A_2, \tag{33}
$$

with $P_A(0) = Q$, converges to the unique positive definite solution to the anti-Riccati equation (24).

Very recently, we have studied a class of nonlinear matrix equations in the form of[16,17]

$$
X + A^{\mathrm{H}} X^{-\#} A = I_n, \tag{34}
$$

where $A \in \mathbf{C}^{n\times n}$ is known. Next, we show how to link the anti-Riccati equation (24) with this class of nonlinear matrix equations. To this end, we define

$$Q_0 = Q^{-1} + \left(A_2 Q^{-1} A_2^{\mathrm{H}}\right)^{\#} + \left(B_2 R^{-1} B_2^{\mathrm{H}}\right)^{\#} > 0. \tag{35}$$

Proposition 1. *If the anti-Riccati equation (24) has a positive definite solution P_A, then the nonlinear matrix equation (34), with*

$$A = Q_0^{-\frac{\#}{2}} A_2 Q^{-1} Q_0^{-\frac{1}{2}}, \tag{36}$$

also has a positive definite solution X such that

$$X = Q_0^{-\frac{1}{2}} \left(P_A^{-1} + \left(A_2 Q^{-1} A_2^{\mathrm{H}}\right)^{\#} + \left(B_2 R^{-1} B_2^{\mathrm{H}}\right)^{\#}\right) Q_0^{-\frac{1}{2}}. \tag{37}$$

Moreover, if the antilinear system (3) is stabilizable, then X given by (37) is the maximal solution to (34).

Proof. The proof of this lemma is similar to the case of normal discrete-time Riccati equation (see, for example, the work of Adam and Assimakis[19]). By using (A4), we get from (24) that

$$-Q + P_A = A_2^{\mathrm{H}} \left(P_A^{-\#} + B_2 R^{-1} B_2^{\mathrm{H}}\right)^{-1} A_2.$$

Let

$$P_A^{-\#} + B_2 R^{-1} B_2^{\mathrm{H}} = T_1. \tag{38}$$

Then, we have $P_A = A_2^{\mathrm{H}} T_1^{-1} A_2 + Q$. Taking inverse on both sides of this equation to give

$$\begin{aligned} P_A^{-1} &= \left(A_2^{\mathrm{H}} T_1^{-1} A_2 + Q\right)^{-1} \\ &= Q^{-1} - Q^{-1} A_2^{\mathrm{H}} \left(T_1 + A_2 Q^{-1} A_2^{\mathrm{H}}\right)^{-1} A_2 Q^{-1}, \end{aligned}$$

where we have used (A4) again. Let

$$T_1 + A_2 Q^{-1} A_2^{\mathrm{H}} = T_2^{\#}. \tag{39}$$

Then, in view of (38), we further have

$$\begin{aligned} Q^{-1} - Q^{-1} A_2^{\mathrm{H}} T_2^{-\#} A_2 Q^{-1} &= P_A^{-1} \\ &= T_1^{\#} - \left(B_2 R^{-1} B_2^{\mathrm{H}}\right)^{\#} \\ &= T_2 - \left(A_2 Q^{-1} A_2^{\mathrm{H}}\right)^{\#} - \left(B_2 R^{-1} B_2^{\mathrm{H}}\right)^{\#}, \end{aligned}$$

or equivalently

$$T_2 + Q^{-1} A_2^{\mathrm{H}} T_2^{-\#} A_2 Q^{-1} = Q^{-1} + \left(A_2 Q^{-1} A_2^{\mathrm{H}}\right)^{\#} + \left(B_2 R^{-1} B_2^{\mathrm{H}}\right)^{\#} = Q_0.$$

Because $Q_0 > 0$, we multiply both sides of the above equation on the left and right by $Q_0^{-\frac{1}{2}}$ to give

$$Q_0^{-\frac{1}{2}} T_2 Q_0^{-\frac{1}{2}} + Q_0^{-\frac{1}{2}} Q^{-1} A_2^{\mathrm{H}} Q_0^{-\frac{\#}{2}} \left(Q_0^{-\frac{1}{2}} T_2 Q_0^{-\frac{1}{2}}\right)^{-\#} Q_0^{-\frac{\#}{2}} A_2 Q^{-1} Q_0^{-\frac{1}{2}} = I_n. \tag{40}$$

With A defined in (36), we have

$$A^{\mathrm{H}} = \left(Q_0^{-\frac{1}{2}}\right)^{\mathrm{H}} \left(Q^{-1}\right)^{\mathrm{H}} A_2^{\mathrm{H}} \left(Q_0^{-\frac{\#}{2}}\right)^{\mathrm{H}} = Q_0^{-\frac{1}{2}} Q^{-1} A_2^{\mathrm{H}} Q_0^{-\frac{\#}{2}}.$$

Then, (40) is just (34) by setting

$$
\begin{aligned}
X &\triangleq Q_0^{-\frac{1}{2}} T_2 Q_0^{-\frac{1}{2}} \\
&= Q_0^{-\frac{1}{2}} \left(T_1^{\#} + \left(A_2 Q^{-1} A_2^{\mathrm{H}} \right)^{\#} \right) Q_0^{-\frac{1}{2}} \\
&= Q_0^{-\frac{1}{2}} \left(P_A^{-1} + \left(A_2 Q^{-1} A_2^{\mathrm{H}} \right)^{\#} + \left(B_2 R^{-1} B_2^{\mathrm{H}} \right)^{\#} \right) Q_0^{-\frac{1}{2}},
\end{aligned}
$$

which is (37).

Let (34) have another positive definite solution $X_1 \geq X$. Then, there exists a positive definite matrix $P_1 \leq P_A$ such that

$$
X_1 = Q_0^{-\frac{1}{2}} \left(P_1^{-1} + \left(B_2 R^{-1} B_2^{\mathrm{H}} \right)^{\#} + \left(A_2 Q^{-1} A_2^{\mathrm{H}} \right)^{\#} \right) Q_0^{-\frac{1}{2}}.
$$

Reversing the procedure in the above shows that P_1 is also a positive definite solution to the anti-Riccati equation (24). This leads to a contradiction as the positive definite solution to (24) is unique when system (3) is stabilizable. □

By this proposition, when system (3) is stabilizable, the unique positive solution to the anti-Riccati equation (24) can be obtained by computing the maximal solution to the nonlinear matrix equation (34), which has been carefully studied in our other works.[16,17] We finally remark that, as indicated by Proposition 1, if system (3) is stabilizable and (34) has any other positive definite solutions X_2, then we must have

$$
X_2 \leq Q_0^{-\frac{1}{2}} \left(P_A^{-1} + \left(B_2 R^{-1} B_2^{\mathrm{H}} \right)^{\#} + \left(A_2 Q^{-1} A_2^{\mathrm{H}} \right)^{\#} \right) Q_0^{-\frac{1}{2}}.
$$

3.2 | A normal Riccati equation–based approach

In this section, we establish a normal Riccati equation–based approach to the LQR problem for the antilinear system (3). For notation simplicity, we denote

$$
\begin{cases}
A_N = A_2^{\#} \left(I_n - B_2 \left(R + B_2^{\mathrm{H}} Q^{\#} B_2 \right)^{-1} B_2^{\mathrm{H}} Q^{\#} \right) A_2, \\
B_N = \left[B_2^{\#} \; A_2^{\#} B_2 \right], \\
Q_N = Q + A_2^{\mathrm{H}} \left(Q^{-\#} + B_2 R^{-1} B_2^{\mathrm{H}} \right)^{-1} A_2, \\
R_N = \begin{bmatrix} R^{\#} & 0 \\ 0 & R + B_2^{\mathrm{H}} Q^{\#} B_2 \end{bmatrix}.
\end{cases}
\tag{41}
$$

Theorem 3. *Consider the antilinear system (3). Then, the following three statements are equivalent:*

1. *The LQR problem associated with system (3) has a solution.*
2. *The system (3) is stabilizable, namely, (6) is satisfied.*
3. *There is a unique positive definite solution P_N to the normal Riccati equation*

$$
-Q_N = A_N^{\mathrm{H}} P_N A_N - P_N - A_N^{\mathrm{H}} P_N B_N \left(R_N + B_N^{\mathrm{H}} P_N B_N \right)^{-1} B_N^{\mathrm{H}} P_N A_N. \tag{42}
$$

In this case, the optimal controller is the normal-state feedback (7) with $K_1 = K_1^*$ defined by

$$
K_1^* = -\left(\left(R + B_2^{\mathrm{H}} Q^{\#} B_2 \right)^{-1} B_2^{\mathrm{H}} Q^{\#} A_2 + [0 \quad I_m] \left(R_N + B_N^{\mathrm{H}} P_N B_N \right)^{-1} B_N^{\mathrm{H}} P_N A_N \right), \tag{43}
$$

the closed-loop system is asymptotically stable, and the minimal value of $J(u)$ is

$$
J_{\min}(u) = x_0^{\mathrm{H}} P_N x_0. \tag{44}
$$

Proof. Clearly, we only need to show the equivalence of (2) and (3). Denote

$$A_L = A_2^{\#}A_2,\ B_L = B_N,\ R_L = R_N,\ Q_L = Q + A_2^{\mathrm{H}}Q^{\#}A_2,\ L_L = \begin{bmatrix} 0_{n\times m} & A_2^{\mathrm{H}}Q^{\#}B_2 \end{bmatrix}.$$

We use system (3) repeatedly to give

$$x(k+2) = A_2^{\#}A_2x(k) + A_2^{\#}B_2u(k) + B_2^{\#}u^{\#}(k+1). \tag{45}$$

Let $k = 2\tau$, $\xi(\tau) = x(2\tau) = x(k)$, and

$$v(\tau) = \begin{bmatrix} u^{\#}(2\tau+1) \\ u(2\tau) \end{bmatrix} = \begin{bmatrix} u^{\#}(k+1) \\ u(k) \end{bmatrix}. \tag{46}$$

Then, system (45) can be further equivalently expressed by

$$\xi(\tau+1) = A_L\xi(\tau) + B_Lv(\tau), \tag{47}$$

where $\tau \geq 0$, and the initial condition is $\xi(0) = x(0) = x_0$. Notice that (47) is a normal linear system with the same dimension as system (3) (yet the dimension for the input has been doubled).

Now, consider the functional (8). In view of (3), we have

$$\begin{aligned} J(u) &= \sum_{\tau=0}^{\infty}\left(x^{\mathrm{H}}(2\tau)Qx(2\tau) + x^{\mathrm{H}}(2\tau+1)Qx(2\tau+1) + u^{\mathrm{H}}(2\tau)Ru(2\tau) + u^{\mathrm{H}}(2\tau+1)Ru(2\tau+1)\right) \\ &= \sum_{\tau=0}^{\infty}\left(x^{\mathrm{H}}(2\tau)Qx(2\tau) + x^{\#\mathrm{H}}(2\tau+1)Q^{\#}x^{\#}(2\tau+1) + u^{\mathrm{H}}(2\tau)Ru(2\tau) + u^{\#\mathrm{H}}(2\tau+1)R^{\#}u^{\#}(2\tau+1)\right) \\ &= \sum_{\tau=0}^{\infty}\Big(x^{\mathrm{H}}(2\tau)Qx(2\tau) + (A_2x(2\tau) + B_2u(2\tau))^{\mathrm{H}}Q^{\#}\left(A_2x(2\tau) + B_2u(2\tau)\right) \\ &\quad + u^{\mathrm{H}}(2\tau)Ru(2\tau) + u^{\#\mathrm{H}}(2\tau+1)R^{\#}u^{\#}(2\tau+1)\Big) \\ &= \sum_{\tau=0}^{\infty}\begin{bmatrix} x(2\tau) \\ u^{\#}(2\tau+1) \\ u(2\tau) \end{bmatrix}^{\mathrm{H}}\begin{bmatrix} Q + A_2^{\mathrm{H}}Q^{\#}A_2 & 0 & A_2^{\mathrm{H}}Q^{\#}B_2 \\ 0 & R^{\#} & 0 \\ B_2^{\mathrm{H}}Q^{\#}A_2 & 0 & R + B_2^{\mathrm{H}}Q^{\#}B_2 \end{bmatrix}\begin{bmatrix} x(2\tau) \\ u^{\#}(2\tau+1) \\ u(2\tau) \end{bmatrix} \\ &= \sum_{\tau=0}^{\infty}\begin{bmatrix} \xi(\tau) \\ v(\tau) \end{bmatrix}^{\mathrm{H}}\begin{bmatrix} Q_L & L_L \\ L_L^{\mathrm{H}} & R_L \end{bmatrix}\begin{bmatrix} \xi(\tau) \\ v(\tau) \end{bmatrix}, \end{aligned} \tag{48}$$

where $\xi(\tau)$ and $v(\tau)$ are those in system (47). Design a preliminary state feedback

$$v(\tau) = K_0\xi(\tau) + w(\tau), \tag{49}$$

for system (47), where

$$K_0 = -R_L^{-1}L_L^{\mathrm{H}} = -\begin{bmatrix} 0 \\ \left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}A_2 \end{bmatrix}.$$

Then, we can show that

$$
\begin{aligned}
&\begin{bmatrix} \xi(\tau) \\ v(\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} Q_L & L_L \\ L_L^{\mathrm{H}} & R_L \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ v(\tau) \end{bmatrix} \\
&= \begin{bmatrix} \xi(\tau) \\ K_0\xi(\tau) + w(\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} Q_L & L_L \\ L_L^{\mathrm{H}} & R_L \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ K_0\xi(\tau) + w(\tau) \end{bmatrix} \\
&= \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} I_n & K_0^{\mathrm{H}} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} Q_L & L_L \\ L_L^{\mathrm{H}} & R_L \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K_0 & I_m \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix} \\
&= \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} Q_L + K_0^{\mathrm{H}} R_L K_0 + L_L K_0 + K_0^{\mathrm{H}} L_L^{\mathrm{H}} & L_L + K_0^{\mathrm{H}} R_L \\ L_L^{\mathrm{H}} + R_L K_0 & R_L \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix} \\
&= \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} Q_L + K_0^{\mathrm{H}} R_L K_0 + L_L K_0 + K_0^{\mathrm{H}} L_L^{\mathrm{H}} & 0 \\ 0 & R_L \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ w(\tau) \end{bmatrix}.
\end{aligned}
$$

Direct computation gives

$$
\begin{aligned}
&Q_L + K_0^{\mathrm{H}} R_L K_0 + L_L K_0 + K_0^{\mathrm{H}} L_L^{\mathrm{H}} \\
&= Q_L - L_L R_L^{-1} L_L^{\mathrm{H}} \\
&= Q + A_2^{\mathrm{H}} Q^{\#} A_2 - A_2^{\mathrm{H}} Q^{\#} B_2 \left(R + B_2^{\mathrm{H}} Q^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} Q^{\#} A_2 \\
&= Q_N = Q + A_2^{\mathrm{H}} \left(Q^{-\#} + B_2 R^{-1} B_2^{\mathrm{H}}\right)^{-1} A_2 \\
&> Q > 0.
\end{aligned}
$$

Consequently, the index function (48) becomes

$$
J(u) = \sum_{\tau=0}^{\infty} \left(\xi^{\mathrm{H}}(\tau) Q_N \xi(\tau) + w^{\mathrm{H}}(\tau) R_N w(\tau)\right). \tag{50}
$$

On the other hand, with the preliminary state feedback (49), the linear system (3) or system (47) becomes

$$
\begin{aligned}
\xi(\tau+1) &= (A_L + B_L K_0)\xi(\tau) + B_L w(\tau) \\
&= \left(A_2^{\#} A_2 - A_2^{\#} B_2 \left(R + B_2^{\mathrm{H}} Q^{\#} B_2\right)^{-1} B_2^{\mathrm{H}} Q^{\#} A_2\right) \xi(\tau) + B_L w(\tau) \\
&= A_N \xi(\tau) + B_N w(\tau), \ \ \xi(0) = x(0) = x_0.
\end{aligned}
$$

This is a normal linear system with a normal quadratic index function (50) in the same form as (8). Hence, by the standard LQR theory,[7,9] the optimal control problem has a solution if and only if (A_N, B_N) is stabilizable, which is further equivalent to the existence of a unique positive definite solution to the normal Riccati equation (42). As $(A_N, B_N) = (A_L + B_L K_0, B_L)$, (A_N, B_N) is stabilizable if and only if (A_L, B_L) is stabilizable,[4] namely, by Corollary 1, the antilinear system (3) is stabilizable. This proves the equivalence of (2) and (3).

When the normal Riccati equation (42) has a (unique) positive definite solution, the optimal control is just the state feedback $w(\tau) = K_N \xi(\tau)$ with

$$
K_N = -\left(R_N + B_N^{\mathrm{H}} P_N B_N\right)^{-1} B_N^{\mathrm{H}} P_N A_N.
$$

Notice that

$$\begin{bmatrix} u^{\#}(k+1) \\ u(k) \end{bmatrix} = v(\tau) = K_0\xi(\tau) + w(\tau)$$
$$= K_0\xi(\tau) + K_N\xi(\tau)$$
$$= K_0x(k) + K_Nx(k)$$
$$= -\begin{bmatrix} 0 \\ \left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}A_2 \end{bmatrix} x(k) + K_Nx(k), \tag{51}$$

by which

$$u(k) = \left(-\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}A_2 + \begin{bmatrix} 0 & I_m \end{bmatrix} K_N\right) x(k)$$
$$= K_1x(k). \tag{52}$$

This proves (43). Finally, by the standard LQR theory,[7,9] the optimal controller is also the stabilizing controller and the minimal value of $J(u)$ is

$$J_{\min}(u) = \xi^{\mathrm{H}}(0)P_N\xi(0) = x_0^{\mathrm{H}}P_Nx_0.$$

The proof is finished. □

For system (3), Theorem 3 is better than Lemma 2 in the sense that the dimension of the Riccati equation in Theorem 3 is half of that in Lemma 2, and Theorem 3 is better than Theorem 2 in the sense that a normal Riccati equation is involved in Theorem 3 while a nonstandard Riccati equation is involved in Theorem 2.

Remark 2. The closed-loop system with the optimal control determined by Theorem 3 is

$$x(k+1) = \left(A_2^{\#} + B_2^{\#}K_1^{\#}\right) x^{\#}(k),$$

by which

$$u(k+1) = K_1x(k+1) = K_1\left(A_2^{\#} + B_2^{\#}K_1^{\#}\right) x^{\#}(k). \tag{53}$$

On the other hand, it follows from (51) that

$$u(k+1) = \begin{bmatrix} I_m & 0 \end{bmatrix} K_N^{\#}x^{\#}(k). \tag{54}$$

This and (53) imply

$$K_1\left(A_2^{\#} + B_2^{\#}K_1^{\#}\right) = \begin{bmatrix} I_m & 0 \end{bmatrix} K_N^{\#}.$$

This identity, though is determined implicitly by Theorem 3, is however not easy to show by manipulation.

3.3 | Relationships among three Riccati equations

The following theorem links solutions to these three different Riccati equations (9), (24), and (42).

Theorem 4. *Consider the antilinear system (3). Then, the following three statements are equivalent:*

1. *The bimatrix Riccati equation (9) has a unique positive definite solution* $\{P_1, P_2\}$.
2. *The anti-Riccati equation (24) has a unique positive definite solution* P_A.
3. *The normal Riccati equation (42) has a unique positive definite solution* P_N.

Moreover, these solutions satisfy $P_2 = 0$ *and*

$$P_1 = P_A = P_N. \tag{55}$$

Proof. In view of Lemma 2, Theorem 1, and Theorem 3, we need only to show that $P_A = P_N$. We consider the iteration (33) for the anti-Riccati equation (24) and show that $P_1(k)$ is monotonically increasing, namely,

$$P_A(k+1) \geq P_A(k) \geq Q,\ k \geq 0. \tag{56}$$

Similarly to the proof of Theorem 1, we show this by mathematical induction. Notice that

$$\begin{aligned} P_A(1) &= A_2^{\mathrm{H}}\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2 + Q \\ &= Q_N \geq Q = P_A(0). \end{aligned} \tag{57}$$

Therefore, (56) is true with $k = 0$. We assume that (56) is true with $k = i$, namely, $P_A(i+1) \geq P_A(i) \geq Q > 0$, which implies that $P_A^{-1}(i+1) \leq P_A^{-1}(i)$ or $P_A^{-\#}(i+1) + B_2R^{-1}B_2^{\mathrm{H}} \leq P_A^{-1}(i) + B_2R^{-1}B_2^{\mathrm{H}}$, namely,

$$\left(P_A^{-\#}(i+1) + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} \geq \left(P_A^{-1}(i) + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}.$$

Then, for $k = i+1$, we have

$$\begin{aligned} &P_A(i+2) - P_A(i+1) \\ &= A_2^{\mathrm{H}}\left(\left(P_A^{-\#}(i+1) + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} - \left(P_A^{-\#}(i) + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}\right)A_2 \\ &\geq 0, \end{aligned}$$

which indicates that (56) is true with $k = i+1$.

Now, using (33) repeatedly gives

$$\begin{aligned} P_A(k+2) &= A_2^{\mathrm{H}}\left(P_A^{-\#}(k+1) + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2 + Q \\ &= A_2^{\mathrm{H}}\left(\left(A_2^{\mathrm{T}}\left(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}}\right)^{-1}A_2^{\#} + Q^{\#}\right)^{-1} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2 + Q. \end{aligned} \tag{58}$$

Notice that, by using the identity (A4) twice,

$$\begin{aligned} &\left(\left(A_2^{\mathrm{T}}\left(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}}\right)^{-1}A_2^{\#} + Q^{\#}\right)^{-1} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} \\ =&\left(Q^{-\#} - Q^{-\#}A_2^{\mathrm{T}}\left(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}Q^{-\#}A_2^{\mathrm{T}}\right)^{-1}A_2^{\#}Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} \\ =&\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}} - Q^{-\#}A_2^{\mathrm{T}}\left(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}Q^{-\#}A_2^{\mathrm{T}}\right)^{-1}A_2^{\#}Q^{-\#}\right)^{-1} \\ =&\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} + \left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}Q^{-\#}A_2^{\mathrm{T}}(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}Q^{-\#}A_2^{\mathrm{T}} \\ &- A_2^{\#}Q^{-\#}\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}Q^{-\#}A_2^{\mathrm{T}})^{-1}A_2^{\#}Q^{-\#}\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} \\ =&\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} + \left(I_n + Q^{\#}B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2^{\mathrm{T}}(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}Q^{-\#}A_2^{\mathrm{T}} \\ &- A_2^{\#}Q^{-\#}\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}Q^{-\#}A_2^{\mathrm{T}})^{-1}A_2^{\#}\left(I_n + B_2R^{-1}B_2^{\mathrm{H}}Q^{\#}\right)^{-1} \\ =&\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} + \left(I_n + Q^{\#}B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2^{\mathrm{T}}(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}Q^{-\#}A_2^{\mathrm{T}} \\ &- A_2^{\#}Q^{-\#}\left(Q^{\#} - Q^{\#}B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}\right)Q^{-\#}A_2^{\mathrm{T}})^{-1}A_2^{\#}\left(I_n + B_2R^{-1}B_2^{\mathrm{H}}Q^{\#}\right)^{-1} \\ =&\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1} + \left(I_n + Q^{\#}B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2^{\mathrm{T}}(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}} \\ &+ A_2^{\#}B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}A_2^{\mathrm{T}})^{-1}A_2^{\#}\left(I_n + B_2R^{-1}B_2^{\mathrm{H}}Q^{\#}\right)^{-1}. \end{aligned} \tag{59}$$

With the notation B_N and R_N in (41), we have

$$B_2^{\#}R^{-\#}B_2^{\mathrm{T}} + A_2^{\#}B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}A_2^{\mathrm{T}} = B_N R_N^{-1} B_N^{\mathrm{H}}. \tag{60}$$

By using the identity (A4) again, we obtain

$$\left(I_n + B_2R^{-1}B_2^{\mathrm{H}}Q^{\#}\right)^{-1} = I_n - B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}. \tag{61}$$

Therefore, by substituting (59), (60), and (61) into (58), it yields

$$\begin{aligned}
P_A(k+2) &= A_2^{\mathrm{H}}\left(\left(A_2^{\mathrm{T}}\left(P_A^{-1}(k) + B_2^{\#}R^{-\#}B_2^{\mathrm{T}}\right)^{-1}A_2^{\#} + Q^{\#}\right)^{-1} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2 + Q \\
&= Q + A_2^{\mathrm{H}}\left(Q^{-\#} + B_2R^{-1}B_2^{\mathrm{H}}\right)^{-1}A_2 + A_2^{\mathrm{H}}\left(I_n - B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}\right)A_2^{\mathrm{T}}\left(P_A^{-1}(k)\right. \\
&\quad \left. + B_NR_N^{-1}B_N^{\mathrm{H}}\right)^{-1}A_2^{\#}\left(I_n - B_2\left(R + B_2^{\mathrm{H}}Q^{\#}B_2\right)^{-1}B_2^{\mathrm{H}}Q^{\#}\right)A_2 \\
&= Q_N + A_N^{\mathrm{H}}\left(P_A^{-1}(k) + B_NR_N^{-1}B_N^{\mathrm{H}}\right)^{-1}A_N.
\end{aligned} \tag{62}$$

Now, because the antilinear system (3) is stabilizable, by Theorem 3, for the normal Riccati equation (42), we can construct the iteration

$$P_N(k+1) = Q_N + A_N^{\mathrm{H}}\left(P_N^{-1}(k) + B_NR_N^{-1}B_N^{\mathrm{H}}\right)^{-1}A_N, \tag{63}$$

with $P_N(0) = Q_N$. Then, similarly to the proof of Theorem 1 (see, for example, the work of Assimakis et al[18]),

$$\lim_{k\to\infty} P_N(k) = P_N. \tag{64}$$

Thus, in view of (62) and (63), we can see that

$$P_N(k) = P_A(2k+1),\ k \geq 0. \tag{65}$$

As a result, we get from (64) that

$$\lim_{k\to\infty} P_A(2k+1) = \lim_{k\to\infty} P_N(k) = P_N. \tag{66}$$

On the other hand, by using (56) and (65) again, we have

$$P_N(k-1) = P_A(2k-1) \leq P_A(2k) \leq P_A(2k+1) = P_N(k).$$

Therefore, it follows from (64) that

$$P_N = \lim_{k\to\infty} P_N(k-1) \leq \lim_{k\to\infty} P_A(2k) \leq \lim_{k\to\infty} P_N(k) = P_N. \tag{67}$$

Combining (66) and (67) gives

$$P_A \triangleq \lim_{k\to\infty} P_A(k) = P_N.$$

This completes the proof. □

As a by-product of the above proof, we see from (65) that the iteration (63) for the normal Riccati equation (42) converges faster than the iteration (33) for the anti-Riccati equation (24). Thus, from the computational point of view, the normal Riccati equation (42) is recommended to use.

Remark 3. It follows from this theorem that the bimatrix Riccati equation–based optimal gain (11), the anti-Riccati equation–based optimal gain (26), and the normal Riccati equation–based optimal gain (43) are equivalent.

Remark 4. Another simple proof for $P_A = P_N$ can be given as follows. By Theorems 2 and 3, the optimal problem is solvable with respectively the minimal value $J_{\min}(u) = x_0^{\mathrm{H}} P_A x_0$ and $J_{\min}(u) = x_0^{\mathrm{H}} P_N x_0$. As both P_A and P_N are independent of x_0, we must have

$$x_0^{\mathrm{H}} P_A x_0 = x_0^{\mathrm{H}} P_N x_0, \ \forall x_0 \in \mathbf{C}^n. \tag{68}$$

Next, we claim that, for positive definite matrices P_A and P_N, $P_A = P_N$ if and only if (68). Clearly, we need only to prove the "if" part. Denote $P_A = [a_{ij}]$ and $P_N = [n_{ij}], i,j \in \mathbf{I}[1,n]$. Letting $x_0 = e_i$, where e_i is the ith column of I_n, in (68) gives $a_{ii} = n_{ii}, i \in \mathbf{I}[1,n]$. Letting $x_0 = [1, a + \mathrm{j}b, 0, \ldots, 0]^{\mathrm{H}}$, where $a \in \mathbf{R}, b \in \mathbf{R}$, in (68) gives

$$a\,\mathrm{Re}(a_{12}) - b\,\mathrm{Im}(a_{12}) = a\,\mathrm{Re}(n_{12}) - b\,\mathrm{Im}(n_{12}),$$

which, by respectively choosing ($a = 0, b \neq 0$) and ($b = 0, a \neq 0$), implies respectively $\mathrm{Im}(a_{12}) = \mathrm{Im}(n_{12})$ and $\mathrm{Re}(a_{12}) = \mathrm{Re}(n_{12})$, namely, $a_{12} = n_{12}$. Similarly, if we choose $x_0 = [1, 0, a + \mathrm{j}b, 0, \ldots, 0]^{\mathrm{H}}$, we get $a_{13} = n_{13}$. Repeating this process, we finally have $P_A = P_N$. However, the current proof for Theorem 4 has its own value because it reveals the relationship (see Equation (65)) between the iteration (33) for the anti-Riccati equation (24) and the iteration (63) for the normal Riccati equation (42).

4 | APPLICATIONS TO OPTIMAL CONTROL OF TIME-DELAY SYSTEMS

4.1 | System and problem descriptions

In this section, we consider the following discrete-time time-delay system with only one-step delay[20]

$$\xi(k+1) = A_0\xi(k) + A_{\mathrm{d}}\xi(k-1) + Gv(k), \ k \geq 0, \tag{69}$$

where $A_0, A_{\mathrm{d}} \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times p}$ are known matrices, $\xi \in \mathbf{R}^n$ is the state vector, and $v \in \mathbf{R}^p$ is the control vector. The initial condition is $\xi(0) \in \mathbf{R}^n$ and $\xi(-1) \in \mathbf{R}^n$. Without loss of generality, we assume that $p = 2m$, namely, p is an even number. Otherwise, we let $v = [v^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$ and $G = [G, 0]$ where w is any slack variable. Thus, we can let

$$G = [G_1 \ \ G_2], v(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}, G_i \in \mathbf{R}^{n\times m}, v_i \in \mathbf{R}^m, i = 1, 2. \tag{70}$$

The problem to be solved is finding $v(k)$ for system (69) such that the following quadratic index function is minimized

$$J(v) = \sum_{k=0}^{\infty} \left(\xi^{\mathrm{T}}(k)Q_0\xi(k) + v^{\mathrm{T}}(k)R_0 v(k)\right), \tag{71}$$

where $Q_0 \in \mathbf{R}^{n\times n}$ and $R_0 \in \mathbf{R}^{m\times m}$ are given positive definite matrices.

Remark 5. We explain that we can assume without loss of generality that R_0 is a block diagonal matrix. Denote

$$R_0 = \begin{bmatrix} R_{01} & R_{02} \\ R_{02}^{\mathrm{T}} & R_{03} \end{bmatrix}, \ R_{0i} \in \mathbf{R}^{m\times m}, i = 1, 2, 3.$$

Because $R_0 > 0$, by the Schur complement, we have $R_{03} - R_{02}^{\mathrm{T}} R_{01}^{-1} R_{02} > 0$. Thus, we can denote

$$L_0 = \begin{bmatrix} I_m & -R_{01}^{-1} R_{02} \left(R_{03} - R_{02}^{\mathrm{T}} R_{01}^{-1} R_{02}\right)^{-\frac{1}{2}} R_{01}^{\frac{1}{2}} \\ 0 & \left(R_{03} - R_{02}^{\mathrm{T}} R_{01}^{-1} R_{02}\right)^{-\frac{1}{2}} R_{01}^{\frac{1}{2}} \end{bmatrix}.$$

Direct computation gives

$$L_0^{\mathrm{T}} R_0 L_0 = \begin{bmatrix} R_{01} & \\ & R_{01} \end{bmatrix} > 0.$$

Then, by the input transformation $\hat{v} = L_0^{-1} v$, the time-delay system (69) can be written as

$$\xi(k+1) = A_0 \xi(k) + A_{\mathrm{d}} \xi(k-1) + \hat{G} \hat{v}(k),$$

where $\hat{G} = G L_0$, and the quadratic index function (71) becomes

$$\begin{aligned} J(v) &= \sum_{k=0}^{\infty} \left(\xi^{\mathrm{T}}(k) Q_0 \xi(k) + \hat{v}^{\mathrm{T}}(k) L_0^{\mathrm{T}} R_0 L_0 \hat{v}(k)\right) \\ &= \sum_{k=0}^{\infty} \left(\xi^{\mathrm{T}}(k) Q_0 \xi(k) + \hat{v}^{\mathrm{T}}(k) \begin{bmatrix} R_{01} & \\ & R_{01} \end{bmatrix} \hat{v}(k)\right). \end{aligned}$$

Thus, without loss of generality, we can assume that

$$R_0 = \begin{bmatrix} R & \\ & R \end{bmatrix}, \ 0 < R \in \mathbf{R}^{m \times m}. \tag{72}$$

Therefore we assume hereafter that R_0 takes the special form (72).

Proposition 2. *The time-delay system (69) can be equivalently written as (1), where*

$$\begin{cases} x(k) = \xi(k) + \mathrm{j}\xi(k-1), \\ u(k) = v_1(k) + \mathrm{j} v_2(k), \ k \geq 0, \end{cases} \tag{73}$$

and

$$\begin{cases} A_1 = \frac{1}{2} A_0 + \frac{\mathrm{j}}{2}(I_n - A_{\mathrm{d}}), \ B_1 = \frac{1}{2} G_1 - \frac{\mathrm{j}}{2} G_2, \\ A_2 = \frac{1}{2} A_0 - \frac{\mathrm{j}}{2}(I_n + A_{\mathrm{d}}), \ B_2 = \frac{1}{2} G_1 - \frac{\mathrm{j}}{2} G_2. \end{cases} \tag{74}$$

Moreover, if R_0 takes the form (72), the quadratic index function (71) can be written as

$$J_1(u) = \sum_{k=0}^{\infty} \left(x^{\mathrm{H}}(k) Q x(k) + u^{\mathrm{H}}(k) R u(k)\right) - \xi^{\mathrm{T}}(-1) Q \xi(-1), \tag{75}$$

where $Q = \frac{1}{2} Q_0 > 0$.

Proof. With the notations defined in this proposition, we can compute

$$
\begin{aligned}
&\{A_1, A_2\}x(k) + \{B_1, B_2\}u(k) \\
&= A_1 x(k) + A_2^{\#} x^{\#}(k) + B_1 u(k) + B_2^{\#} u^{\#}(k) \\
&= \left(\frac{1}{2}A_0 + \frac{\mathrm{j}}{2}(I_n - A_\mathrm{d})\right)(\xi(k) + \mathrm{j}\xi(k-1)) + \left(\frac{1}{2}A_0 + \frac{\mathrm{j}}{2}(I_n + A_\mathrm{d})\right)(\xi(k) - \mathrm{j}\xi(k-1)) \\
&\quad + \left(\frac{1}{2}G_1 - \frac{\mathrm{j}}{2}G_2\right)(v_1(k) + \mathrm{j}v_2(k)) + \left(\frac{1}{2}G_1 + \frac{\mathrm{j}}{2}G_2\right)(v_1(k) - \mathrm{j}v_2(k)) \\
&= \left(\frac{1}{2}A_0 + \frac{1}{2}A_0\right)\xi(k) + \left(\frac{1}{2}(I_n + A_\mathrm{d}) - \frac{1}{2}(I_n - A_\mathrm{d})\right)\xi(k-1) \\
&\quad + \mathrm{j}\left(\frac{1}{2}A_0\xi(k-1) + \frac{1}{2}(I_n - A_\mathrm{d})\xi(k) - \frac{1}{2}A_0\xi(k-1) + \frac{1}{2}(I_n + A_\mathrm{d})\xi(k)\right) \\
&\quad + \left(\frac{1}{2}G_1 + \frac{1}{2}G_1\right)v_1(k) + \left(\frac{1}{2}G_2 + \frac{1}{2}G_2\right)v_2(k) \\
&\quad + \mathrm{j}\left(\frac{1}{2}G_1 v_2(k) - \frac{1}{2}G_2 v_1(k) - \frac{1}{2}G_1 v_2(k) + \frac{1}{2}G_2 v_1(k)\right) \\
&= A_0\xi(k) + A_\mathrm{d}\xi(k-1) + Gv(k) + \mathrm{j}\xi(k) \\
&= \xi(k+1) + \mathrm{j}\xi(k) \\
&= x(k+1),
\end{aligned}
$$

which shows that $x(k)$ satisfies (1). Notice that

$$
\begin{aligned}
J(v) &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\xi^{\mathrm{T}}(k)Q_0\xi(k) + \sum_{k=0}^{\infty}\xi^{\mathrm{T}}(k)Q_0\xi(k)\right) + \sum_{k=0}^{\infty}v^{\mathrm{T}}(k)R_0 v(k) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}\xi^{\mathrm{T}}(k-1)Q_0\xi(k-1) + \sum_{k=0}^{\infty}\xi^{\mathrm{T}}(k)Q_0\xi(k)\right) + \sum_{k=0}^{\infty}v^{\mathrm{T}}(k)R_0 v(k) - \frac{1}{2}\xi^{\mathrm{T}}(-1)Q_0\xi(-1) \\
&= \sum_{k=0}^{\infty}\begin{bmatrix}\xi(k)\\ \xi(k-1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\frac{1}{2}Q_0 & \\ & \frac{1}{2}Q_0\end{bmatrix}\begin{bmatrix}\xi(k)\\ \xi(k-1)\end{bmatrix} + \sum_{k=0}^{\infty}v^{\mathrm{T}}(k)R_0 v(k) - \frac{1}{2}\xi^{\mathrm{T}}(-1)Q_0\xi(-1).
\end{aligned} \tag{76}
$$

On the other hand, for any $Q_x = Q_1 + \mathrm{j}Q_2 > 0$, where $Q_i, i = 1, 2$ are real matrices, we can compute

$$
\begin{aligned}
x^{\mathrm{H}}(k)Q_x x(k) &= \left(\xi^{\mathrm{T}}(k) - \mathrm{j}\xi^{\mathrm{T}}(k-1)\right)(Q_1 + \mathrm{j}Q_2)(\xi(k) + \mathrm{j}\xi(k-1)) \\
&= \left(\xi^{\mathrm{T}}(k)Q_1 + \xi^{\mathrm{T}}(k-1)Q_2\right)\xi(k) - \left(\xi^{\mathrm{T}}(k)Q_2 - \xi^{\mathrm{T}}(k-1)Q_1\right)\xi(k-1) \\
&\quad + \left(\left(\xi^{\mathrm{T}}(k)Q_1 + \xi^{\mathrm{T}}(k-1)Q_2\right)\xi(k-1) + \left(\xi^{\mathrm{T}}(k)Q_2 - \xi^{\mathrm{T}}(k-1)Q_1\right)\xi(k)\right)\mathrm{j} \\
&= \left(\xi^{\mathrm{T}}(k)Q_1 + \xi^{\mathrm{T}}(k-1)Q_2\right)\xi(k) - \left(\xi^{\mathrm{T}}(k)Q_2 - \xi^{\mathrm{T}}(k-1)Q_1\right)\xi(k-1) \\
&= \begin{bmatrix}\xi(k)\\ \xi(k-1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q_1 & -Q_2\\ Q_2 & Q_1\end{bmatrix}\begin{bmatrix}\xi(k)\\ \xi(k-1)\end{bmatrix},
\end{aligned}
$$

where we have noticed that $Q_1 = Q_1^{\mathrm{T}}$ and $Q_2 = -Q_2^{\mathrm{T}}$. Similarly, for any $R_u = R_1 + \mathrm{j}R_2 > 0$, where $R_i, i = 1, 2$, are real matrices, we have

$$
u^{\mathrm{H}}(k)R_u u(k) = \begin{bmatrix}v_1(k)\\ v_2(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R_1 & -R_2\\ R_2 & R_1\end{bmatrix}\begin{bmatrix}v_1(k)\\ v_2(k)\end{bmatrix} = v^{\mathrm{T}}(k)\begin{bmatrix}R_1 & -R_2\\ R_2 & R_1\end{bmatrix}v(k).
$$

Thus, in view of (72), it yields from (76) that, if we denote $Q_x = \frac{1}{2}Q_0$ and $R_u = R$, the quadratic index function $J(v)$ can be exactly written as (75). The proof is finished. □

Since the last term $\xi^{\mathrm{T}}(-1)Q\xi(-1)$ in (75) dependents on only the initial condition, $J_1(u)$ is minimized if and only if

$$J_2(u) = \sum_{k=0}^{\infty} \left(x^{\mathrm{H}}(k)Qx(k) + u^{\mathrm{H}}(k)Ru(k)\right) \tag{77}$$

is minimized. Hence, the linear optimal control problem for the time-delay system (69) has been transformed equivalently to the linear quadratic optimal control problem for the complex-valued linear system (1) with the quadratic index function (77). According to results in Section 2, the solution to this problem has been completely characterized by Lemma 2. Thus, the optimal control is $u(k) = K_1^* x(k) + \left(K_2^*\right)^{\#} x^{\#}$, which, by separating real and imaginary parts, is equivalent to our other work[1]:

$$v(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\left(K_1^* + K_2^*\right) & -\mathrm{Im}\left(K_1^* + K_2^*\right) \\ \mathrm{Im}\left(K_1^* - K_2^*\right) & \mathrm{Re}\left(K_1^* - K_2^*\right) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \xi(k-1) \end{bmatrix},$$

which is physically implementable.[1]

4.2 | An illustrative example

In this section, we use the linearized F-16 aircraft model studied previously in the works of Sobel and Shapiro[21] and Liu and Zhou[22] to illustrate the obtained results. The continuous-time model is shown as follows

$$\dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{A}_{\mathrm{d}}\xi(t-\tau) + \mathcal{G}v(t), \tag{78}$$

in which we have assumed that there is a state delay $\tau = 0.1$ in the elevator deflection, which is the fourth element of $x(t)$.[22] The coefficient matrices are then given by[22]

$$\mathcal{A} = \begin{bmatrix} 0 & 1.0 & 0 & 0 & 0 \\ 0 & -0.8694 & 43.223 & -17.251 & -1.5766 \\ 0 & 0.9934 & -1.3411 & -0.1690 & -0.2518 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20.0 \end{bmatrix},$$

$$\mathcal{A}_{\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20.0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20.0 & 0 \\ 0 & 20.0 \end{bmatrix}.$$

By taking the sampling period as $T = 0.1$ second, the continuous-time time-delay system (78) can be discretized as (69), where

$$A_0 = \begin{bmatrix} 1.0000 & 0.1025 & 0.2080 & -0.0879 & -0.0057 \\ 0 & 1.1175 & 4.1534 & -1.8042 & -0.1010 \\ 0 & 0.0955 & 1.0722 & -0.0994 & -0.0153 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0.1353 \end{bmatrix},$$

$$A_{\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & 0.0594 & 0 \\ 0 & 0 & 0 & -1.8165 & 0 \\ 0 & 0 & 0 & 0.0434 & 0 \\ 0 & 0 & 0 & -2.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} -0.0581 & -0.0040 \\ -1.7586 & -0.1131 \\ -0.0720 & -0.0175 \\ 2.0000 & 0 \\ 0 & 0.8647 \end{bmatrix}.$$

We now consider the corresponding linear optimal control problem (71) with $Q_0 = 2I_5$ and $R = 1$. Thus, $Q = I_5$. Let $\{P_1(k), P_2(k)\}$ be computed according to the iteration (23). Denote $e(k) = \ln \|\{E_1(k), E_2(k)\}\|$, where

$$\{E_1(k), E_2(k)\} = \{A_1, A_2\}^{\mathrm{H}}\left(\{P_1(k), P_2(k)\}^{-1} + \{R_1, R_2\}\right)^{-1}\{A_1, A_2\} + \{Q, 0\} - \{P_1(k), P_2(k)\}.$$

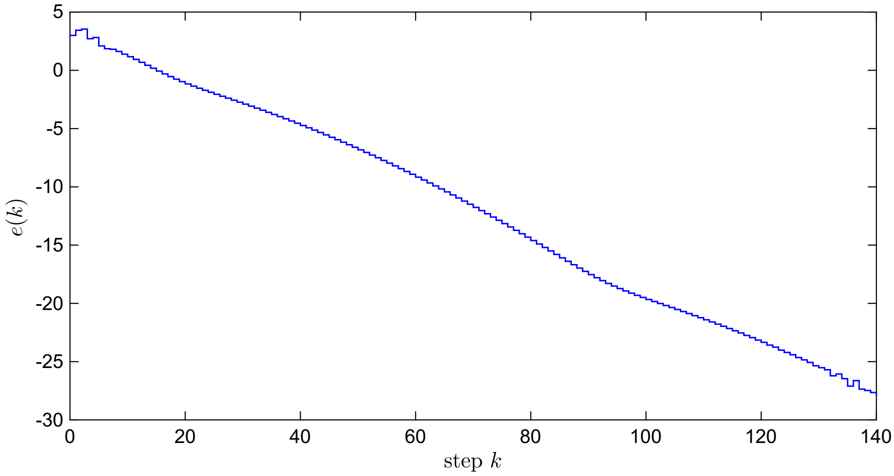

FIGURE 1 The iteration error $e(k)$ for (23) [Colour figure can be viewed at wileyonlinelibrary.com]

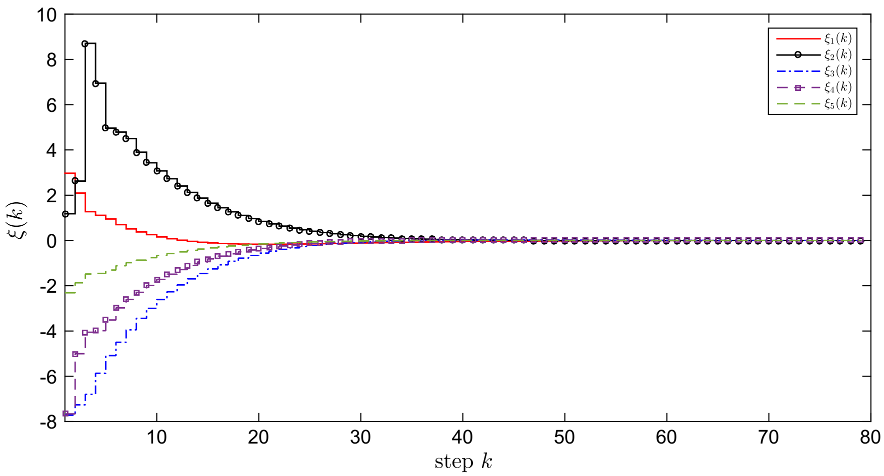

FIGURE 2 State responses of the closed-loop system [Colour figure can be viewed at wileyonlinelibrary.com]

The history of $e(k)$ as a function of k is plotted in Figure 1 from which we can see that it converges to zero in a rather fast speed. For $k = 140$, we obtain

$$P_1(k) = \begin{bmatrix} 12.3464 + 0.0000j & 2.3671 + 0.0000j & 8.6342 + 0.0000j & -2.0194 + 3.2919j & -0.1762 + 0.0000j \\ 2.3671 + 0.0000j & 3.7958 + 0.0000j & 10.5820 + 0.0000j & -3.6990 + 5.5020j & -0.2025 + 0.0000j \\ 8.6342 + 0.0000j & 10.5820 + 0.0000j & 56.7625 + 0.0000j & -18.9340 + 23.6474j & -0.9975 + 0.0000j \\ -2.0194 - 3.2919j & -3.6990 - 5.5020j & -18.9340 - 23.6474j & 28.0046 + 0.0000j & 0.3487 + 0.4725j \\ -0.1762 + 0.0000j & -0.2025 + 0.0000j & -0.9975 + 0.0000j & 0.3487 - 0.4725j & 1.5259 + 0.0000j \end{bmatrix},$$

$$P_2(k) = \begin{bmatrix} 11.3464 + 0.0000j & 2.3671 + 0.0000j & 8.6342 + 0.0000j & -2.0194 + 3.2919j & -0.1762 + 0.0000j \\ 2.3671 + 0.0000j & 2.7958 + 0.0000j & 10.5820 + 0.0000j & -3.6990 + 5.5020j & -0.2025 + 0.0000j \\ 8.6342 + 0.0000j & 10.5820 + 0.0000j & 55.7625 + 0.0000j & -18.9340 + 23.6474j & -0.9975 + 0.0000j \\ -2.0194 + 3.2919j & -3.6990 + 5.5020j & -18.9340 + 23.6474j & -3.5897 - 17.1002j & 0.3487 - 0.4725j \\ -0.1762 + 0.0000j & -0.2025 + 0.0000j & -0.9975 + 0.0000j & 0.3487 - 0.4725j & 0.5259 + 0.0000j \end{bmatrix}.$$

It follows that $P_1(k) = P_1^{\mathrm{H}}(k)$ and $P_2(k) = P_2^{\mathrm{T}}(k)$. Consequently, the optimal feedback gain $\{K_1^*, K_2^*\}$ can be computed according to (11) as

$$K_1^* = \begin{bmatrix} 0.0463 + 0.0962j & 0.1140 + 0.1205j & 0.6384 + 0.6279j & -0.7529 - 0.4637j & -0.0112 - 0.0584j \end{bmatrix},$$
$$K_2^* = \begin{bmatrix} 0.0463 - 0.0962j & 0.1140 - 0.1205j & 0.6384 - 0.6279j & -0.2122 - 0.0036j & -0.0112 + 0.0584j \end{bmatrix}.$$

Finally, for simulation purpose, we choose the initial condition

$$\begin{bmatrix} \xi^{\mathrm{T}}(0) \\ \xi^{\mathrm{T}}(-1) \end{bmatrix} = \begin{bmatrix} 4 & 1 & -8 & -6 & 9 \\ 4 & 4 & 8 & -6 & 10 \end{bmatrix}.$$

The state trajectories of the closed-loop system are recorded in Figure 2 from which we can observe the asymptotic stability of the closed-loop system.

5 | CONCLUSION

This paper has studied linear optimal control (LQR) of discrete-time complex-valued linear systems. Firstly, an iterative algorithm was proposed to solve the associated bimatrix Riccati equation introduced in our early study. The convergence of the algorithm was proven. Then, the LQR problem for the antilinear system, which is a special case of the complex-valued linear system, was carefully studied and three different solutions were obtained, namely, bimatrix Riccati equation–based solution, anti-Riccati equation–based solution, and normal Riccati equation–based solution. Relationships among these three different solutions are revealed. The bimatrix Riccati equation–based approach was then used to solve the LQR problem of linear time-delay systems with one-step state delay, and an illustrative example demonstrated the effectiveness of the proposed approach.

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APPENDIX

BIMATRIX AND ITS PROPERTIES

Let $(\{A_1, A_2\} + \{B_1, B_2\})x = \{A_1, A_2\}x + \{B_1, B_2\}x$ and $\{A_1, A_2\}\{B_1, B_2\}x = \{A_1, A_2\}(\{B_1, B_2\}x)$. We denote $\{A_1, A_2\} = \{B_1, B_2\} \in \{\mathbf{C}^{n\times m}, \mathbf{C}^{n\times m}\}$ if $\{A_1, A_2\}x = \{B_1, B_2\}x, \forall x \in \mathbf{C}^m$. Then, the bimatrix has some properties[1]:

$$\begin{aligned}\{A_1, A_2\}\{B_1, B_2\} &= \{A_1B_1 + A_2^{\#}B_2, A_1^{\#}B_2 + A_2B_1\},\\ \{A_1, A_2\} + \{A_3, A_4\} &= \{A_1 + A_3, A_2 + A_4\},\end{aligned} \tag{A1}$$

and $(\{A_1, A_2\}\{A_3, A_4\})\{A_5, A_6\} = \{A_1, A_2\}(\{A_3, A_4\}\{A_5, A_6\})$. We say that $\{A_1, A_2\} \in \{\mathbf{C}^{n\times m}, \mathbf{C}^{n\times m}\}$ is a zero bimatrix (denoted by $\mathcal{O}_{n\times m}$) if $\{A_1, A_2\}x = 0_{n\times 1}$ for any $x \in \mathbf{C}^m$. If $n = m$, we say that $\{A_1, A_2\}$ is a square bimatrix. A square bimatrix $\{A_1, A_2\}$ is an identity bimatrix (denoted by $\mathcal{I}_n$) if $\{A_1, A_2\}x = x, \forall x \in \mathbf{C}^n$. We have shown that[1]

$$\begin{aligned}\{A_1, A_2\} &= \mathcal{I}_n \iff (A_1, A_2) = (I_n, 0_{n\times n}),\\ \{A_1, A_2\} &= \mathcal{O}_{n\times m} \iff (A_1, A_2) = (0_{n\times m}, 0_{n\times m}).\end{aligned} \tag{A2}$$

The power of the square bimatrix $\{A_1, A_2\}$, denoted by $\{A_1, A_2\}^i$ with $i \in \mathbf{Z}^+$, can be defined recursively as $\{A_1, A_2\}^i = \{A_1, A_2\}\{A_1, A_2\}^{i-1}$ with $\{A_1, A_2\}^0 = \mathcal{I}_n$, according to (A1).

For a square bimatrix $\{A_1, A_2\}$, if there exists another square bimatrix $\{A_3, A_4\}$ such that $\{A_1, A_2\}\{A_3, A_4\} = \{A_3, A_4\}\{A_1, A_2\} = \mathcal{I}_n$, then $\{A_3, A_4\}$ is called the inverse bimatrix of $\{A_1, A_2\}$ and is denoted by $\{A_3, A_4\} = \{A_1, A_2\}^{-1}$. A square bimatrix $\{A_1, A_2\}$ is invertible if and only if

$$\{A_1, A_2\}_{\triangleq} \triangleq \begin{bmatrix} A_1 & A_2^{\#} \\ A_2 & A_1^{\#} \end{bmatrix}$$

is invertible.[1] Moreover, $\{A_1, A_2\}^{-1} = \{A_3, A_4\}$, where[1]

$$\begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = (\{A_1, A_2\}_{\triangleq})^{-1} \begin{bmatrix} I_n \\ 0_{n\times n} \end{bmatrix}.$$

Therefore, $\{A_1, 0_{n\times n}\}$ is invertible if and only if A_1 is nonsingular, and $\{0_{n\times n}, A_2\}$ is invertible if and only if A_2 is nonsingular. Moreover, $\{A_1, 0_{n\times n}\}^{-1} = \{A_1^{-1}, 0_{n\times n}\}$ and $\{0_{n\times n}, A_2\}^{-1} = \{0_{n\times n}, A_2^{-\#}\}$.[1] From (A2), the norm of a bimatrix $\{A_1, A_2\} \in \{\mathbf{C}^{n\times m}, \mathbf{C}^{n\times m}\}$ can be defined as

$$\|\{A_1, A_2\}\| = \|A_1\| + \|A_2\|.$$

The conjugate-transpose of the bimatrix $\{A_1, A_2\} \in \{\mathbf{C}^{n\times m}, \mathbf{C}^{n\times m}\}$ is defined as[1]

$$\{A_1, A_2\}^{\mathrm{H}} \triangleq \left\{A_1^{\mathrm{H}}, A_2^{\mathrm{T}}\right\} = \left\{A_1^{\mathrm{H}}, A_2^{\#\mathrm{H}}\right\}.$$

The square bimatrix $\{P_1, P_2\} \in \{\mathbf{C}^{n\times n}, \mathbf{C}^{n\times n}\}$ is said to be Hermite if[1]

$$\{P_1, P_2\} = \{P_1, P_2\}^{\mathrm{H}} = \left\{P_1^{\mathrm{H}}, P_2^{\mathrm{T}}\right\}. \tag{A3}$$

A bimatrix $\{P_1, P_2\}$ is said to be (semi)positive definite (denoted by $\{P_1, P_2\} > (\geq)0$), if it is a square Hermite bimatrix and, for any $x \in \mathbf{C}^n$, $\mathrm{Re}(x^{\mathrm{H}}\{P_1, P_2\}x) > (\geq)0, \forall x \neq 0$.[1] It has been shown in our other work[1] that $\{P_1, P_2\}$ is (semi)positive definite if and only if $\{P_1, P_2\}_\diamond > (\geq)0$. It can be readily shown that $\{P_1, P_2\} \geq \{Q_1, Q_2\} > 0$ implies $\{P_1, P_2\}^{-1} \leq \{Q_1, Q_2\}^{-1}$, and, for any $\{A_1, A_2\}$,

$$\{A_1, A_2\}^{\mathrm{H}}\{P_1, P_2\}\{A_1, A_2\} \geq \{A_1, A_2\}^{\mathrm{H}}\{Q_1, Q_2\}\{A_1, A_2\}.$$

For given matrices A, B, C, D with appropriate dimensions and A, C and $C^{-1} + DA^{-1}B$ are invertible, the well-known Sherman-Morrison-Woodbury formula holds true[23]

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1}DA^{-1}. \tag{A4}$$

The following lemma extends this identity to the bimatrix case. The proof is straightforward and is omitted.

Lemma 3. *Let $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$, and $\{D_1, D_2\}$ be some given bimatrices with appropriate dimensions. Assume that $\{A_1, A_2\}$, $\{C_1, C_2\}$ and $\{C_1, C_2\}^{-1} - \{D_1, D_2\}\{A_1, A_2\}^{-1}\{B_1, B_2\}$ are all invertible. Then,*

$$\begin{aligned}
&(\{A_1, A_2\} + \{B_1, B_2\}\{C_1, C_2\}\{D_1, D_2\})^{-1} \\
=&\{A_1, A_2\}^{-1} - \{A_1, A_2\}^{-1}\{B_1, B_2\}(\{C_1, C_2\}^{-1} \\
&+ \{D_1, D_2\}\{A_1, A_2\}^{-1}\{B_1, B_2\})^{-1}\{D_1, D_2\}\{A_1, A_2\}^{-1}.
\end{aligned}$$