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SPECTRAL RADIUS AND AVERAGE 2-DEGREE SEQUENCE OF A GRAPH

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In a simple connected graph, the average 2-degree of a vertex is the average degree of its neighbors. With the average 2-degree sequence and the maximum degree ratio of adjacent vertices, we present a sharp upper bound of the spectral radius of the adjacency matrix of a graph, which improves a result in [Y. H. Chen, R. Y. Pan and X. D. Zhang, Two sharp upper bounds for the signless Laplacian spectral radius of graphs, *Discrete Math. Algorithms Appl.* **3**(2) (2011) 185–191].

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$, edge set $E(G)$ and $n = |V(G)|$. For any vertex $v \in V(G)$, let d_v denote the degree of v, define the *average 2-degree* M_v of v to be the average degree of the neighbors of v. In other words, $M_v = \sum_{u\sim v} d_u/d_v$, where $u \sim v$ means vertices u and v are adjacent. Throughout the paper, label the vertices of G by $1, 2, \ldots, n$ such that $M_1 \geq M_2 \geq \cdots \geq M_n$. Let $A = (a_{ij})$ be the *adjacency matrix* of G, a binary matrix of order n such that for any pair $i, j \in V(G)$, $a_{ij} = 1$ if and only if i, j are adjacent in G. The *spectral radius* $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix; this parameter has been studied by many authors [1, 2, 5–9, 11–13] and can be used to induce some other bounds such as the upper bounds of signless Laplacian eigenvalues [3, 4].

The following theorem is well-known and referred as Perron–Frobenius Theorem [10, Chap. 2].

Theorem 1.1. *If B is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(B)$ and row-sums $r_1, r_2, \ldots, r_n$, then*

$$\rho(B) \leq \max_{1 \leq i \leq n} r_i$$

with equality if and only if the row-sums of B are all equal.

In this paper, we pay attention to the upper bounds of spectral radius of graphs in terms of the average 2-degree sequence. By setting $B = U^{-1}AU$, where $U = \mathrm{diag}$ $(d_1, d_2, \ldots, d_n)$, the following fact is easily seen from Theorem 1.1.

Theorem 1.2.

$$\rho(G) \leq M_1$$

with equality if and only if $M_1 = M_2 = \cdots = M_n$.

A graph for which equality holds in Theorem 1.2 is called *pseudo-regular* in [13].

In 2011 [3, Theorem 2.1], Chen, Pan and Zhang gave the following bound.

Theorem 1.3. *Let $a := \max\{d_i/d_j \mid 1 \leq i, j \leq n\}$. Then*

$$\rho(G) \leq \frac{M_2 - a + \sqrt{(M_2 + a)^2 + 4a(M_1 - M_2)}}{2},$$

with equality if and only if G is pseudo-regular.

We will show in Corollary 3.3 that Theorem 1.3 is indeed a generalization of Theorem 1.2. Moreover, we give the following theorem to generalize Theorem 1.3.

Theorem 1.4. *For any $b \geq \max\{d_i/d_j \mid i \sim j\}$ and $1 \leq \ell \leq n$,*

$$\rho(G) \leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2},$$

with equality if and only if G is pseudo-regular.

Note that Theorem 1.3 is a special case of Theorem 1.4 by taking $b = a$ and $\ell = 2$. The proof of Theorem 1.4 is a subtle application of Perron–Frobenius Theorem. This idea was previously employed in [9, 11]. Indeed, our proof is an edited version of the proof of [9, Theorem 1.7].

We provide some examples of pseudo-regular graphs that are not regular in Example 2.1. The lowest upper bound among the choices of b and ℓ is investigated in Sec. 3.

2. Proof of Theorem 1.4

Proof of Theorem 1.4. For each $1 \leq i \leq \ell - 1$, let $x_i \geq 1$ be a variable to be determined later. Let $U = \mathrm{diag}(d_1 x_1, \ldots, d_{\ell-1}x_{\ell-1}, d_\ell, \ldots, d_n)$ be a diagonal matrix of size $n \times n$. Consider the matrix $B = U^{-1}AU$. Note that A and B have the same eigenvalues. Let $r_1, r_2, \ldots, r_n$ be the row-sums of B. Then for $1 \leq i \leq \ell - 1$ we have

$$\begin{aligned} r_i &= \sum_{k=1}^{\ell-1} \frac{1}{d_i x_i} a_{ik} d_k x_k + \sum_{k=\ell}^{n} \frac{1}{d_i x_i} a_{ik} d_k \\ &= \frac{1}{x_i} \sum_{k=1}^{\ell-1} (x_k - 1) a_{ik} \frac{d_k}{d_i} + \frac{1}{x_i} \sum_{k=1}^{n} a_{ik} \frac{d_k}{d_i} \\ &\leq \frac{b}{x_i} \left(\sum_{k=1, k\neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i, \end{aligned} \tag{2.1}$$

since $a_{ik} d_k / d_i \leq b$. Similarly for $\ell \leq j \leq n$ we have

$$\begin{aligned} r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk} \frac{d_k}{d_j} + \sum_{k=\ell}^{n} a_{jk} \frac{d_k}{d_j} \\ &= \sum_{k=1}^{\ell-1} (x_k - 1) a_{jk} \frac{d_k}{d_j} + \sum_{k=1}^{n} a_{jk} \frac{d_k}{d_j} \\ &\leq b \left(\sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) + M_\ell. \end{aligned} \tag{2.2}$$

Let

$$\phi_\ell = \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2}.$$

For $1 \leq i \leq \ell - 1$ let

$$x_i = 1 + \frac{M_i - M_\ell}{\phi_\ell + b} \geq 1. \tag{2.3}$$

Then for $1 \leq i \leq \ell - 1$ we have

$$\begin{aligned} r_i &\leq \frac{b}{x_i} \left(\sum_{k=1, k\neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i \\ &= \frac{b \sum_{k=1}^{\ell-1} (M_k - M_\ell) + \phi_\ell M_i + b M_\ell}{\phi_\ell + b + M_i - M_\ell} \\ &= \frac{\frac{1}{4}[(M_\ell - b)^2 + (M_\ell + b)^2 + 4b \sum_{k=1}^{\ell-1} (M_k - M_\ell) - 2M_\ell^2 - 2b^2 + 4bM_\ell] + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell} \end{aligned}$$

$$= \frac{\phi_\ell^2 + \phi_\ell b - \phi_\ell M_\ell + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell}$$
$$= \phi_\ell.$$

For $\ell \leq j \leq n$ we have

$$r_i \leq b\left(\sum_{k=1}^{\ell-1} x_k - (\ell-1)\right) + M_\ell$$
$$= \frac{b\sum_{k=1}^{\ell-1}(M_k - M_\ell) + \phi_\ell M_\ell + bM_\ell}{\phi_\ell + b}$$
$$= \frac{\frac{1}{4}[4b\sum_{k=1}^{\ell-1}(M_k - M_\ell) + 2M_\ell\sqrt{(M_\ell+b)^2 + 4b\sum_{k=1}^{\ell-1}(M_k - M_\ell)} + 2M_\ell^2 + 2bM_\ell]}{\phi_\ell + b}$$
$$= \frac{\phi_\ell^2 + \phi_\ell b}{\phi_\ell + b}$$
$$= \phi_\ell.$$

Hence by Theorem 1.1,

$$\rho(G) = \rho(B) \leq \max_{1\leq i\leq n}\{r_i\} \leq \phi_\ell.$$

The first part of Theorem 1.4 follows.

Suppose $M_1 = M_2 = \cdots = M_n$. Then $\rho(G) = M_1 = \phi_\ell$ by Theorem 1.2. Hence the equality in Theorem 1.4 follows.

To prove the necessary condition, suppose $\rho(G) = \phi_\ell$. Applying Theorem 1.1 and the inequalities in (2.1) and (2.2), $\phi_\ell = \rho(G) \leq \max_{1\leq i\leq n} r_i \leq \phi_\ell$. Hence $r_1 = r_2 = \cdots = r_n = \phi_\ell$, and the equalities in (2.1) and (2.2) hold. In particular,

$$b = a_{ik}\frac{d_k}{d_i} \tag{2.4}$$

for any $1 \leq i \leq n$ and $1 \leq k \leq \ell - 1$ with $x_k - 1 > 0$, and $M_\ell = M_n$. We consider three cases:

(1) Suppose $M_1 = M_\ell$: Clearly $M_1 = M_n$.

(2) $M_{t-1} > M_t = M_\ell$ for some $3 \leq t \leq \ell$: Then $x_k > 1$ for $1 \leq k \leq t-1$ by (2.3). Hence by (2.4)

$$b = a_{12}\frac{d_2}{d_1} = a_{21}\frac{d_1}{d_2} = 1,$$

and $d_i = n - 1$ for all $i = 1, 2, \ldots, n$. This implies that G is regular, a contradiction.

(3) $M_1 > M_2 = M_\ell$: Then $x_1 > 1$ by (2.3). Hence by (2.4), $b = a_{i1}d_1/d_i$ for $2 \leq i \leq n$. Hence $d_1 = n-1$ and $d_2 = d_3 = \cdots = d_n = (n-1)/b$. Then $(n-1)/b = M_1 > M_2 = M_n = (n-1)/b - 1 + b$. This implies $b < 1$, a contradiction.

This completes the proof of the theorem. □

Example 2.1. The graphs in Figs. 1–3 are pseudo-regular but not regular. In particular, the graph in Fig. 3 has a cycle C_k of k vertices, and shares each vertex of C_k with a triangle K_3.

An interesting problem could be characterizing all the nonregular pseudo-regular graphs.

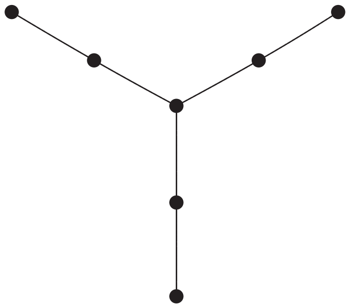

Fig. 1. Graph with $M_i = 2$.

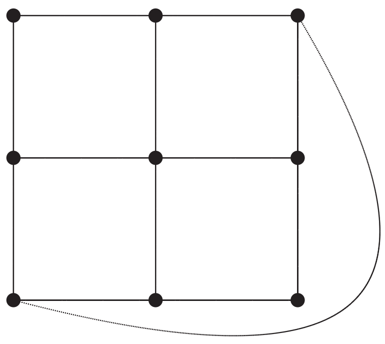

Fig. 2. Graph with $M_i = 3$.

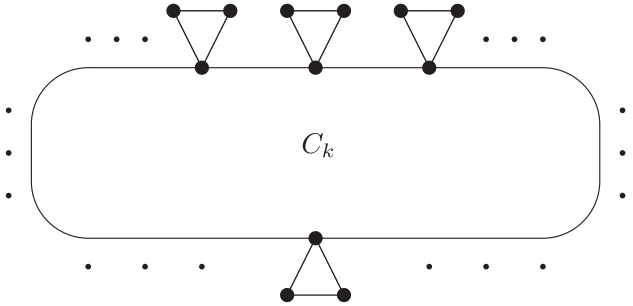

Fig. 3. Graphs with $M_i = 3$.

3. The Shape of the Sequence $\phi_1, \phi_2, \ldots, \phi_n$

Given a decreasing sequence $M_1 \geq M_2 \geq \cdots \geq M_n$ of positive integers, consider the functions

$$\phi_\ell(x) = \frac{M_\ell - x + \sqrt{(M_\ell + x)^2 + 4x\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2}$$

for $x \in [1, \infty)$. Note that $\phi_\ell(b)$ is the upper bound of $\rho(G)$ in Theorem 1.4.

The following proposition shows that the smaller the b in Theorem 1.4 is, the lower the upper bound of $\rho(G)$ reaches.

Proposition 3.1. *For any $1 \leq \ell \leq n$, $\phi_\ell(x)$ is increasing on $[1, \infty)$.*

Proof. For convenience, let

$$S = \sum_{i=1}^{\ell-1}(M_i - M_\ell).$$

To show that $\phi_\ell(x)$ is increasing on $[1, \infty)$, it is sufficient to show that the derivative of $\phi_\ell(x)$ is nonnegative. This follows from the following equivalent steps.

$$\begin{aligned}
& \phi_\ell'(x) \geq 0 \\
\Leftrightarrow \quad & -1 + \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} \geq 0 \\
\Leftrightarrow \quad & \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} \geq 1 \\
\Leftrightarrow \quad & (M_\ell + x + 2S)^2 \geq (M_\ell + x)^2 + 4Sx \\
\Leftrightarrow \quad & 4SM_\ell + 4S^2 \geq 0.
\end{aligned}$$

□

Note that for $1 \leq s \leq n-1$, $M_s = M_{s+1}$ implies $\phi_s(x) = \phi_{s+1}(x)$. We adopt the same viewpoint as [9, Proposition 3.1] to describe when the bound gets improved throughout the sequence $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ in the following proposition.

Proposition 3.2. *Suppose $M_s > M_{s+1}$ for some $1 \leq s \leq n-1$, and let the symbol $\succeq$ denote $>$ or $=$. Then*

$$\phi_s(x) \succeq \phi_{s+1}(x) \quad \textit{if and only if} \; \sum_{i=1}^{s} M_i \succeq xs(s-1).$$

Proof. Consider the following equivalent relations step by step.

$$\begin{aligned}
& \phi_s(x) > \phi_{s+1}(x) \\
\Leftrightarrow \quad & M_s - M_{s+1} + \sqrt{(M_s + x)^2 + 4x\sum_{i=1}^{s-1}(M_i - M_s)}
\end{aligned}$$

$$> \sqrt{(M_{s+1}+x)^2 + 4x\sum_{i=1}^{s}(M_i - M_{s+1})}$$

$$\Leftrightarrow \quad \sqrt{(M_s+x)^2 + 4x\sum_{i=1}^{s-1}(M_i - M_s)} > 2xs - (M_s + x)$$

$$\Leftrightarrow \quad (M_s+x)^2 + 4x\sum_{i=1}^{s}(M_i - M_s) > 4x^2s^2 - 4xs(M_s+x) + (M_s+x)^2$$

$$\Leftrightarrow \quad \sum_{i=1}^{s} M_i > xs(s-1),$$

where the third relation is obtained from the second by taking square on both sides, simplifying it, and deleting the common term $M_s - M_{s+1}$. Note that even if $2xs - (M_s + x) < 0$ in the third relation, squaring both sides would be proper since then $\sqrt{(M_s+x)^2 + 4x\sum_{i=1}^{s-1}(M_i - M_s)} \geq |M_s + x| \geq |2xs - (M_s + x)|$. Similarly, note that if $\sum_{i=1}^{s} M_i = xs(s-1)$, then $M_s \leq xs$ and $2xs - (M_s + x) \geq 0$. Hence

$$\phi_s(x) = \phi_{s+1}(x) \tag{3.1}$$

$$\Leftrightarrow \quad \sqrt{(M_s+x)^2 + 4x\sum_{i=1}^{s-1}(M_i - M_s)} = 2xs - (M_s + x)$$

$$\Leftrightarrow \quad (M_s+x)^2 + 4x\sum_{i=1}^{s}(M_i - M_s) = 4x^2s^2 - 4xs(M_s+x) + (M_s+x)^2$$

$$\Leftrightarrow \quad \sum_{i=1}^{s} M_i = xs(s-1). \qquad \square$$

The following corollary shows that Theorem 1.3 is an improvement of Theorem 1.2.

Corollary 3.3. *For any $x \in [1, \infty)$, $\phi_2(x) \leq M_1$ with equality if and only if $M_2 = M_1$.*

Proof. If $M_2 = M_1$ then $\phi_2(x) = M_2 \leq M_1$. Suppose $M_2 < M_1$. Choose $s = 1$ and the symbol $\succeq$ to be $>$ in Proposition 3.2,

$$M_1 = \phi_1(x) > \phi_2(x). \qquad \square$$

Choosing $b = \max\{d_i/d_j \mid i \sim j\}$, by Proposition 3.2 with $s = 2$ and $x = b$, if $M_2 > M_3$ and $M_1 + M_2 > 2b$, then $\phi_2(b) > \phi_3(b)$. This is a case when Theorem 1.4 is truly an improvement of Theorem 1.3.

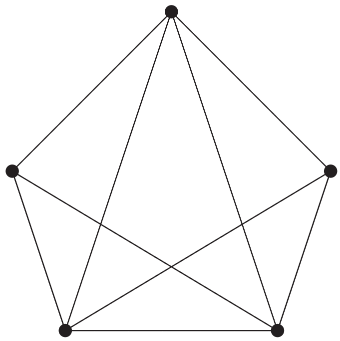

Fig. 4. Graph with $\phi_2 > \phi_3$.

Example 3.4. For the graph in Fig. 4, $M_1 = M_2 = 4$, $M_3 = 7/2$, $b = 4/3$, $\phi_1(b) = \phi_2(b) = 4$, $\phi_3(b) \doteqdot 3.762$ and $\rho(G) = 1 + \sqrt{7} \doteqdot 3.646$.

Note that $\phi_1(x) = M_1 \geq \phi_2(x)$ by Corollary 3.3, and for $2 \leq t \leq n-1$, $\sum_{i=1}^{t} M_i < xt(t-1)$ implies $M_t < x(t-1)$, and hence $\sum_{i=1}^{t+1} M_i < xt(t-1) + x(t-1) < xt(t+1)$. This implies that the sequence $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ is composed by two parts. The first part is decreasing and the second part is increasing. In particular, if we choose $x = M_1$, $M_2 > M_3$, $s = 2$ and $\succeq$ to be $>$ in Proposition 3.2, then $M_1 + M_2 \not> 2M_1 = xs(s-1)$, so $\phi_2(M_1) \leq \phi_3(M_1)$. Hence $\phi_2(M_1)$ is smallest among $\phi_1(M_1), \phi_2(M_1), \ldots, \phi_n(M_1)$.

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