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The Comparison of Upper Bounds for Spectral Radius of Weighted Graphs

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Abstract

We consider weighted graphs, where the edge weights are positive definite matrices. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. In this study the upper bounds for the spectral radius of weighted graphs, which edge weights are positive definite matrices, are compared.

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Keywords: Weighted graph, Spectral radius, Adjacency matrix

1 Introduction

We consider simple graphs, that is, graph which have no loops or parallel edges. Hence a graph $G = (V, E)$ consist of a finite set of vertices, V, and a set of edges, E, each of whose elements is an unordered pair of distinct vertices. Generally V is taken as $V = \{1, 2, .., n\}$

A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of same order and will be assumed to be positive matrix. In this paper, by "weighted graph" we will mean "a weighted graph with each of its edges bearing a positive definite matrix as weight", unless otherwise stated.

Now we introduce some notation. Let G be a weighted graph on n vertices. denote by $w_{i,j}$ the positive definite weight matrix of order p of the edge ij, and assume that $w_{i,j} = w_{j,i}$ .We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j:j\sim i} w_{i,j}$.

The adjacency matrix of a graph G is a block matrix, denoted and defined as $A(G) = (a_{ij})$ where

$$a_{i,j} = \begin{cases} w_{i,j} & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the definition above, the zero denotes the $p \times p$ zero matrix. Thus $A(G)$ is a square matrix of order np. For any symmetric matrix K, let $\rho_1(K)$ denote the largest eigenvalue, in modulus (i.e., the spectral radius), of K.

Let we give some more definitions. Let $G = (V, E)$. If V is the disjoint union of two nonempty sets V_1 and V_2 such that every vertex i in V_1 has the same$\rho_1(w_i)$ and every vertex j in V_2 has the same $\rho_1(w_j)$, then G will be called a weight-semiregular graph. If $\rho_1(w_i) = \rho_1(w_j)$ in weight semiregular graph, then G will be called a weight-regular graph.

Upper and lower bounds for the spectral radius for unweighted graphs have been investigated to a great extent in the literature [2,3,4,5,6,7,8]. In Section 2 of this paper, we give known Lemma and results. Also, we give the upper bounds on the spectral radius of weighted graphs, which is found previously [9,10,11,12]. The main result of this paper, contained in Section 3, gives comparison of the upper bounds whether they are sharper. So we say that a sharper bound is better.

2 The Upper Bounds On The Spectral Radius Of Weighted Graphs

Lemma 1 *(Horn and Johnson [1]) Let B be a Hermitian $n \times n$ matrix with ρ_1 as its largest eigenvalue, in modulus. then for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$,$\bar{y} \in R^n$ $(\bar{y} \neq \bar{0})$, the spectral radius $|\rho_1|$ satisfies*

$$\left|\bar{x}^T B \bar{y}\right| \leq |\rho_1| \sqrt{\bar{x}^T \bar{x}} \sqrt{\bar{y}^T \bar{y}} \tag{1}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of B corresponding to ρ_1 and $\bar{y} = \alpha \bar{x}$ for some $\alpha \in R$.

Lemma 2 *(Weyl, Horn and Johnson [1]) Let $A, B \in M_n$ be Hermitian and let the eigenvalues $\rho_i(A), \rho_i(B)$,and $\rho_i(A + B)$ be arranged in increasşng order $(\rho_n \leq \rho_{n-1} \leq ... \leq \rho_2 \leq \rho_1)$. For each $k = 1, 2, ..., n$ we have*

$$\rho_k(A) + \rho_n(B) \leq \rho_k(A + B) \leq \rho_k(A) + \rho_1(B) \tag{2}$$

Lemma 3 *(Das et al.[9])Let $B_1, B_2, ..., B_k$ be positive definite matrices of order n and let $B = \sum_{i=1}^{n} B_i$. If $\bar{x}$ is an eigenvector of each B_i corresponding* *to the largest eigenvalue $\rho_1(B_i)$ for all i , then $\bar{x}$ is also an eigenvector of B corresponding to the largest eigenvalue $\rho_1(B)$.*

Theorem 4 *(Das et al.[9])Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G ,so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \leq \max_{i\sim j} \left\{ \sqrt{\sum_{k:k\sim i} \rho_1(w_{i,k}) \sum_{k:k\sim j} \rho_1(w_{j,k})} \right\} \tag{3}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (3) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.

Theorem 5 *[10] Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G ,so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \leq \max_{i} \left\{ \sqrt{\sum_{k:k\sim i} \rho_1(w_{i,k}^2) + \sum_{j} \sum_{k:k\in N_i\cap N_j} \rho_1(w_{i,k}w_{k,j})} \right\} \tag{4}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij and $N_i \cap N_j$ is the set of common neighbors of i and j. Moreover equality holds in (4) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.

Corollary 6 *Let G be a simple connected weighted graph in which the edge weights are positive numbers (i.e. 1×1 matrices). Then*

$$\rho_1 \leq \max_{i} \left\{ \sqrt{\sum_{k:k\sim i} w_{i,k}^2 + \sum_{j} \sum_{k:k\in N_i\cap N_j} w_{i,k}w_{k,,j}} \right\} \tag{5}$$

Moreover equality holds in (5) if and only if G is a regular graph or G is a bipartite semiregular graph.

Theorem 7 *[11] Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G ,so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \leq \max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\frac{\rho_1(w_k)}{\rho_1(w_i)}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\rho_1(w_k)}{\rho_1(w_j)}\rho_1(w_{j,k})}\right\} \tag{6}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (6) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.

Theorem 8 *[11] Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G , so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \leq \max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\frac{\alpha_k}{\alpha_i}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\alpha_k}{\alpha_i}\rho_1(w_{j,k})}\right\} \tag{7}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij and $\alpha_i = \sum_{i\sim k}\rho_1(w_{i,k})$ for $1 \leq i \leq n$.Moreover equality holds in (7) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j. $\alpha_i = \sum_{i\sim k}\rho_1(w_{i,k})$.

Corollary 9 *Let G be a simple connected weighted graph in which the edge weights are positive numbers (i.e. 1×1 matrices). Then*

$$\rho_1 \leq \max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\frac{w_k}{w_i}w_{i,k}\sum_{k:k\sim j}\frac{w_k}{w_j}w_{j,k})}\right\} \tag{8}$$

Moreover equality holds in (8) if and only if G is a regular graph or G is a bipartite semiregular graph.

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j. $\alpha_i = \sum_{i\sim k}\rho_1(w_{i,k})$.

Theorem 10 *[12] Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G ,so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \le \max_{i\sim j}\left[\left(\begin{array}{c}\sum\limits_{k:k\sim i}\rho_1(w_{i,k}^2)\\ +\sum\limits_t \sum\limits_{k'\in N_i\cap N_t}\rho_1(w_{i,k'}w_{k',t})\end{array}\right)\left(\begin{array}{c}\sum\limits_{k:k\sim j}\rho_1(w_{j,k}^2)\\ +\sum\limits_t \sum\limits_{k'\in N_j\cap N_t}\rho_1(w_{j,k'}w_{k',t})\end{array}\right)\right]^{\frac{1}{4}} \tag{9}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij and $N_i \cap N_j$ is the set of common neighbors of i and j.

Corollary 11 *Let G be a simple connected weighted graph in which the edge weights are positive numbers (i.e. 1×1 matrices). Then*

$$\rho_1 \le \max_{i\sim j}\left[\left(\begin{array}{c}\sum\limits_{k:k\sim i}w_{i,k}^2\\ +\sum\limits_t \sum\limits_{k'\in N_i\cap N_t}w_{i,k'}w_{k',t}\end{array}\right)\left(\begin{array}{c}\sum\limits_{k:k\sim j}w_{j,k}^2\\ +\sum\limits_t \sum\limits_{k'\in N_j\cap N_t}w_{j,k'}w_{k',t}\end{array}\right)\right]^{\frac{1}{4}} \tag{10}$$

3 The Comparison of These Upper Bounds

Corollary 12 *The upper bound in (6) is better (sharper) than the upper bound in (7)*

Proof. Let the upper bounds in (6) and (7) be maximum for $i \sim j$. We wish prove that

$$\sqrt{\sum_{k:k\sim i}\frac{\rho_1(w_k)}{\rho_1(w_i)}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\rho_1(w_k)}{\rho_1(w_j)}\rho_1(w_{j,k})} \le \sqrt{\sum_{k:k\sim i}\frac{\alpha_k}{\alpha_i}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\alpha_k}{\alpha_i}\rho_1(w_{j,k})}$$

For every $i, j \in V$, since $w_{i,j}$ matrices are Hermitian we get

$$\rho_1(w_k) \le \sum_{t:t\sim k}\rho_1(w_{t,k}) = \alpha_k$$

and

$$\rho_1(w_i) \le \sum_{k:i\sim k}\rho_1(w_{i,k}) = \alpha_i$$

from (2).
i.e.

$$\frac{\rho_1(w_k)}{\rho_1(w_i)} \leq \frac{\alpha_k}{\alpha_i} \tag{11}$$

Multiplying both sides of (11) by $\rho_1(w_{i,k})$ and taking summation over k such that $i \sim k$, we get

$$\sum_{k:k\sim i} \frac{\rho_1(w_k)}{\rho_1(w_i)} \rho_1(w_{i,k}) \leq \sum_{k:k\sim i} \frac{\alpha_k}{\alpha_i} \rho_1(w_{i,k}) \tag{12}$$

Similarly, for j such that $i \sim j$, we have

$$\frac{\rho_1(w_k)}{\rho_1(w_j)} \leq \frac{\alpha_k}{\alpha_j} \tag{13}$$

Multiplying both sides of (13) by $\rho_1(w_{j,k})$ and taking summation over k such that $j \sim k$, we get

$$\sum_{k:k\sim j} \frac{\rho_1(w_k)}{\rho_1(w_j)} \rho_1(w_{j,k}) \leq \sum_{k:k\sim j} \frac{\alpha_k}{\alpha_j} \rho_1(w_{j,k}) \tag{14}$$

From (12) and (14), we get

$$\sum_{k:k\sim i} \frac{\rho_1(w_k)}{\rho_1(w_i)} \rho_1(w_{i,k}) \sum_{k:k\sim j} \frac{\rho_1(w_k)}{\rho_1(w_j)} \rho_1(w_{j,k}) \leq \sum_{k:k\sim i} \frac{\alpha_k}{\alpha_i} \rho_1(w_{i,k}) \sum_{k:k\sim j} \frac{\alpha_k}{\alpha_i} \rho_1(w_{j,k})$$

Hence the corollary is proved. ■

Corollary 13 *The upper bound in (7) is better than the upper bound in (3)*

Proof. Let the upper bounds in (7) and (3) be maximum for $i \sim j$. We wish prove that

$$\sum_{k:k\sim i} \frac{\alpha_k}{\alpha_i} \rho_1(w_{i,k}) \sum_{k:k\sim j} \frac{\alpha_k}{\alpha_i} \rho_1(w_{j,k}) \leq \sum_{k:k\sim i} \rho_1(w_{i,k}) \sum_{k:k\sim j} \rho_1(w_{j,k})$$

We have

$$
\begin{aligned}
\sum_{k:k\sim i}\frac{\alpha_k}{\alpha_i}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\alpha_k}{\alpha_j}\rho_1(w_{j,k}) &\leq \left(\frac{1}{\alpha_i}\sum_{k:k\sim i}\left(\max_{i\sim k}\{\alpha_k\}\right)\rho_1(w_{i,k})\right)\left(\sum_{k:k\sim j}\frac{\alpha_k}{\alpha_j}\rho_1(w_{j,k})\right)\\
&\leq \left(\max_{i\sim k}\{\alpha_k\}\right)\left(\frac{1}{\alpha_j}\sum_{k:k\sim j}\left(\max_{j\sim k}\{\alpha_k\}\right)\rho_1(w_{j,k})\right)\\
&= \left(\max_{i\sim k}\{\alpha_k\}\right)\left(\max_{j\sim k}\{\alpha_k\}\right)\\
&\leq \alpha_i\alpha_j\\
&= \sum_{k:k\sim i}\rho_1(w_{i,k})\sum_{k:k\sim j}\rho_1(w_{j,k})
\end{aligned}
$$

Therefore (7) is better than (3). ■

Corollary 14 *The upper bound in (9) is better than the upper bound in (5)*

Proof. For i vertex of G let the upper bounds in (5) be maximum. So the expression

$$\sum_{k:k\sim i}\rho_1(w_{i,k}^2)+\sum_j\sum_{k:k\in N_i\cap N_j}\rho_1(w_{i,k}w_{k,j}) \tag{15}$$

is maksimum. Let the phrase of (15) say X_i and let the upper bounds in (9) also be maximum for j such that $i\sim j$. So we have

$$Y = X_iX_j \tag{16}$$

Now we wish prove that

$$\sqrt{X_i}\geq\sqrt[4]{X_iX_j} \tag{17}$$

for $i\sim j$.We assume that $X_i = X_j$ for $i\sim j$. Then we get that the equality in (17). So let $X_i\neq X_j$ be. When X_i is maximum for vertex i, we have

$$X_i > X_j \tag{18}$$

for vertex j such that $i\sim j$. From inequality in (15) we get

$$X_iX_i > X_iX_j \tag{19}$$

i.e.

$$\sqrt{X_i}\geq\sqrt[4]{X_iX_j}=\sqrt[4]{Y} \tag{20}$$

Hence (9) is better than (5). ■

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