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# On sharp bounds for spectral radius of nonnegative matrices

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**ABSTRACT**

We give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with positive row sums using average 3-row sums, compare these bounds with the existing bounds using the average 2-row sums by examples, and apply them to the adjacency matrix and the signless Laplacian matrix of a digraph or a graph.

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## 1. Introduction

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. The spectral radius of $A$, denoted by $\rho(A)$, is the largest modulus of the eigenvalues of $A$. It is well known that $\rho(A)$ is an eigenvalue of $A$ (see [1]). The spectral radius of a nonnegative matrix has been studied extensively, see, e.g. [1–8]. In dynamical systems or graph theory, one would like to compute the spectral radius of a nonnegative matrix. For example, the topological entropy, one of the main invariants of a topological dynamical system which tells us how chaotic the system is, can often be computed as a logarithm of the spectral radius of a certain nonnegative matrix.[9]

For $1 \le i \le n$, $r_i(A) = \sum_{j=1}^{n} a_{ij}$ is called the $i$th row sum of $A$, and $M_i(A) = \sum_{j=1}^{n} a_{ij} r_j(A)$ is called the $i$th 2-row sum of $A$. For $1 \le i \le n$ with $r_i(A) > 0$, let $m_i(A) = \frac{M_i(A)}{r_i(A)} = \frac{\sum_{j=1}^{n} a_{ij} r_j(A)}{r_i(A)}$, which is known as the $i$th average 2-row sum of $A$ (see [7]), and let $s_i(A) = \frac{\sum_{j=1}^{n} a_{ij} M_j(A)}{r_i(A)} = \frac{\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a_{jk} r_k(A)}{r_i(A)}$, which we call the $i$th average 3-row sum of $A$ (see [8]). Zhang and Li [8] gave sharp upper and lower bounds for the spectral radius of a nonnegative matrix with positive row sums using maximum and minimum average 3-row sums, respectively, see Lemma 2.3 below.

In this paper, we also consider the spectral radius of some nonnegative matrices associated with a digraph (with no multiple arcs or loops) or a simple graph.

Let $\overrightarrow{G}$ be a digraph with vertex set $V(\overrightarrow{G}) = \{v_1, \dots, v_n\}$. For $v_i, v_j \in V(G)$, the arc from $v_i$ to $v_j$ is denoted by $(v_i, v_j)$, and $v_i$ is called the initial vertex of this arc. Let $d_i^+$ be the out-degree of $v_i$ in $\overrightarrow{G}$, i.e. the number of arcs with initial vertex $v_i$. The adjacency matrix

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of $\overrightarrow{G}$ is the $n \times n$ matrix $A(\overrightarrow{G}) = (a_{ij})$, where $a_{ij} = 1$ if there is an arc from $v_i$ to $v_j$ and 0 otherwise. The signless Laplacian matrix of $\overrightarrow{G}$ is the $n \times n$ matrix $Q(\overrightarrow{G}) = D(\overrightarrow{G}) + A(\overrightarrow{G})$, where $D(\overrightarrow{G})$ is the out-degree diagonal matrix $\mathrm{diag}(d_1^+, \dots, d_n^+)$. The spectral radius of the adjacency matrix of a digraph has been studied extensively, see, e.g. [8,10–12]. The spectral radius of the signless Laplacian matrix of a digraph has been studied in [13].

Let $G$ be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. If we replace each edge $v_iv_j$ of $G$ by two arcs $(v_i, v_j)$ and $(v_j, v_i)$, then we obtain a digraph $\widetilde{G}$. The adjacency matrix and signless Laplacian matrix of $\widetilde{G}$ are called the adjacency matrix and signless Laplacian matrix of $G$, respectively. The spectral radii of the adjacency matrix and the signless Laplacian matrix of a graph have received much attention, see, e.g. [14–17].

In this paper, we give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with positive row sums using average 3-row sums, and characterize the equality cases if the matrix is irreducible. We compare those bounds with the existing bounds using the average 2-row sums by examples, and also apply those bounds to the adjacency matrix and the signless Laplacian matrix of a digraph or a graph.

## 2. Bounds for the spectral radius of nonnegative matrices

We first give several lemmas that will be used.

**Lemma 2.1 [5]:** *Let $A$ be an $n \times n$ nonnegative matrix. Then*

$$\min_{1\le i\le n} r_i(A) \le \rho(A) \le \max_{1\le i\le n} r_i(A).$$

*Moreover, if $A$ is irreducible, then either equality holds if and only if $r_1(A) = \cdots = r_n(A)$.*

For positive integers $s$ and $t$, let $0_{s\times t}$ be the $s \times t$ zero matrix, and let $0_s = 0_{s\times s}$.

**Lemma 2.2 [8]:** *Let $A$ be an $n \times n$ irreducible nonnegative matrix. Then $A^2$ is reducible if and only if there exists a permutation matrix $P$ such that*

$$PAP^\top = \begin{pmatrix} 0_s & A_1 \\ A_2 & 0_{n-s} \end{pmatrix}.$$

*Moreover, $A_2A_1$ and $A_1A_2$ are irreducible, and $\rho(A_1A_2) = \rho(A^2)$.*

**Lemma 2.3 [8]:** *Let $A$ be an $n \times n$ nonnegative matrix with positive row sums. Then*

$$\min_{1\le i\le n} \sqrt{s_i(A)} \le \rho(A) \le \max_{1\le i\le n} \sqrt{s_i(A)}.$$

*Moreover, if $A$ is irreducible, then either equality holds if and only if $m_1(A) = \cdots = m_n(A)$ when $A^2$ is irreducible, and there is a permutation matrix $P$ such that $PAP^\top = \begin{pmatrix} 0_r & A_1 \\ A_2 & 0_{n-r} \end{pmatrix}$, $m_{\sigma(1)}(A) = \cdots = m_{\sigma(r)}(A)$, and $m_{\sigma(r+1)}(A) = \cdots = m_{\sigma(n)}(A)$ when $A^2$ is reducible, where $\sigma$ is a permutation on the set $\{1, \dots, n\}$ which corresponds to the permutation matrix $P$.*

Next we give a sharp upper bound for the spectral radius of a nonnegative matrix.

**Theorem 2.1:** *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $r_1, \dots, r_n$ and with average 3-row sums $s_1 \geq \cdots \geq s_n$. Let $M$ be the largest diagonal element, and $N$ the largest off-diagonal element of $A$. Let $b = \max\left\{\frac{r_i}{r_j} : 1 \leq i, j \leq n\right\}$ and $\theta = M^2 + N^2(n-1) - 2MNb - (n-2)N^2 b$. Suppose that $N > 0$ and $s_1 \geq \theta$ if $b = 1$, and $s_1 > \theta$ if $b > 1$. For $1 \leq l \leq n$, let*

$$\Phi_l = \frac{s_l + \theta + \sqrt{(s_l - \theta)^2 + 4(2MN + (n-2)N^2)b\sum_{k=1}^{l-1}(s_k - s_l)}}{2}.$$

*Then $\rho(A) \leq \sqrt{\Phi_l}$. Moreover, if $A$ is irreducible, then $\rho(A) = \sqrt{\Phi_l}$ for some $1 \leq l \leq n$ if and only if one of the following conditions holds:*

(i) *if $l = 1$, then $m_1(A) = \cdots = m_n(A)$ when $A^2$ is irreducible, and $PAP^\top = \begin{pmatrix} 0_r & A_1 \\ A_2 & 0_{n-r} \end{pmatrix}$ for some permutation matrix $P$ with $m_{\sigma(1)}(A) = \cdots = m_{\sigma(r)}(A)$ and $m_{\sigma(r+1)}(A) = \cdots = m_{\sigma(n)}(A)$ when $A^2$ is reducible, where $\sigma$ is a permutation on the set $\{1, \dots, n\}$ which corresponds to the permutation matrix $P$;*

(ii) *if $2 \leq l \leq n$, then $s_1 = \cdots = s_n$.*

***Proof*:** If $l = 1$, then $\Phi_l = \frac{s_1+\theta+|s_1-\theta|}{2} = s_1$, and thus the result follows immediately from Lemma 2.3.

Suppose that $2 \leq l \leq n$.

If $b = 1$, then $r_1 = \cdots = r_n$, and thus by definition, we have $s_1 = \cdots = s_n$. Since $s_1 \geq \theta$, we have $\Phi_l = \Phi_1 = \frac{s_1+\theta+|s_1-\theta|}{2} = s_1$. By Lemma 2.1, $\rho(A) = r_1 = \sqrt{s_1} = \sqrt{\Phi_l}$.

Suppose that $b > 1$.

Let $U = \mathrm{diag}(x_1 r_1, \dots, x_{l-1} r_{l-1}, r_l, \dots, r_n)$, where $x_i \geq 1$ is a variable to be determined later for $1 \leq i \leq l-1$. Let $B = U^{-1}A^2U$. Obviously, $A^2$ and $B$ have the same eigenvalues. Then $\rho(A) = \sqrt{\rho(A^2)} = \sqrt{\rho(B)}$.

For $1 \leq i \leq l-1$, since $a_{ii} \leq M$, $\frac{r_k}{r_i} \leq b$ for $1 \leq k \leq l-1$ and $k \neq i$, and $a_{ij} \leq N$ for $1 \leq j \leq n$ and $j \neq i$, we have

$$\begin{aligned}
r_i(B) &= r_i(U^{-1}A^2U) \\
&= \frac{1}{r_i x_i}\sum_{k=1}^{l-1} r_k x_k \sum_{j=1}^{n} a_{ij}a_{jk} + \frac{1}{r_i x_i}\sum_{k=l}^{n} r_k \sum_{j=1}^{n} a_{ij}a_{jk} \\
&= \frac{1}{x_i}\left(\sum_{k=1}^{l-1}\sum_{j=1}^{n} a_{ij}a_{jk}\frac{r_k}{r_i}(x_k - 1) + \frac{1}{r_i}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ij}a_{jk}r_k\right) \\
&= \frac{1}{x_i}\left(\sum_{\substack{1\leq k\leq l-1 \\ k\neq i}}\sum_{j=1}^{n} a_{ij}a_{jk}\frac{r_k}{r_i}(x_k - 1) + \sum_{j=1}^{n} a_{ij}a_{ji}(x_i - 1) + \frac{1}{r_i}\sum_{j=1}^{n} a_{ij}\sum_{k=1}^{n} a_{jk}r_k\right) \\
&= \frac{1}{x_i}\left[\sum_{\substack{1\leq k\leq l-1 \\ k\neq i}}\left(a_{ii}a_{ik} + a_{ik}a_{kk} + \sum_{\substack{1\leq j\leq n \\ j\neq i,k}} a_{ij}a_{jk}\right)\frac{r_k}{r_i}(x_k - 1)\right.
\end{aligned}$$

$$
\begin{aligned}
& \left. + \left( a_{ii}a_{ii} + \sum_{\substack{1\le j\le n \\ j\ne i}} a_{ij}a_{ji} \right)(x_i - 1) + s_i \right] \\
\le \; & \frac{1}{x_i}\left( \sum_{\substack{1\le k\le l-1 \\ k\ne i}} \left(2MN + (n-2)N^2\right) b(x_k - 1) + (M^2 + (n-1)N^2)(x_i - 1) + s_i \right) \\
= \; & \frac{1}{x_i}\left( \left(2MN + (n-2)N^2\right) b \sum_{k=1}^{l-1} (x_k - 1) \right. \\
& \left. + (M^2 + (n-1)N^2 - 2MNb - (n-2)N^2 b)(x_i - 1) + s_i \right)
\end{aligned}
$$

with equality when $x_k > 1$ for $1 \le k \le l-1$ only if (a) holds: (a) $a_{kk} = M$, $a_{ij} = N$ for $1 \le j \le n$ with $j \ne i$.

For $l \le i \le n$, since $s_i \le s_l$, $a_{ii} \le M$, $\frac{r_k}{r_i} \le b$ for $1 \le k \le l-1$, and $a_{ij} \le N$ for $1 \le j \le n$ and $j \ne i$, we have

$$
\begin{aligned}
r_i(B) &= r_i(U^{-1}A^2U) \\
&= \frac{1}{r_i}\sum_{k=1}^{l-1}\sum_{j=1}^{n} a_{ij}a_{jk}r_k x_k + \frac{1}{r_i}\sum_{k=l}^{n}\sum_{j=1}^{n} a_{ij}a_{jk}r_k \\
&= \sum_{k=1}^{l-1}\sum_{j=1}^{n} a_{ij}a_{jk}\frac{r_k}{r_i}(x_k - 1) + \sum_{j=1}^{n}\frac{a_{ij}}{r_i}\sum_{k=1}^{n} a_{jk}r_k \\
&= \sum_{k=1}^{l-1}\left( a_{ii}a_{ik} + a_{ik}a_{kk} + \sum_{\substack{1\le j\le n \\ j\ne i,k}} a_{ij}a_{jk} \right)\frac{r_k}{r_i}(x_k - 1) + s_i \\
&\le \sum_{k=1}^{l-1}\left(2MN + (n-2)N^2\right) b(x_k - 1) + s_l \\
&= \left(2MN + (n-2)N^2\right) b \sum_{k=1}^{l-1}(x_k - 1) + s_l
\end{aligned}
$$

with equality when $x_k > 1$ for $1 \le k \le l-1$ only if (b) and (c) hold: (b) $a_{ii} = a_{kk} = M$, $a_{ij} = N$ for $1 \le j \le n$ and $j \ne i$; (c) $s_i = s_l$.

For $1 \le l \le n$, from the expression of $\Phi_l$, we have

$$
\Phi_l^2 - \Phi_l(s_l + \theta) + s_l\theta - (2MN + (n-2)N^2)b\sum_{k=1}^{l-1}(s_k - s_l) = 0,
$$

and thus

$$
(2MN + (n-2)N^2)b\sum_{k=1}^{l-1}(s_k - s_l) = (\Phi_l - s_l)(\Phi_l - \theta).
$$

If $\sum_{k=1}^{l-1}(s_k - s_l) > 0$, then $\Phi_l > \frac{s_l+\theta+|s_l-\theta|}{2} \geq \frac{s_l+\theta-(s_l-\theta)}{2} = \theta$, and otherwise, $s_1 = \cdots = s_l$, and since $s_1 - \theta > 0$, we have $\Phi_l = \Phi_1 = \frac{s_1+\theta+|s_1-\theta|}{2} = s_1 > \theta$. For $1 \leq i \leq l-1$, let $x_i = 1 + \frac{s_i - s_l}{\Phi_l - \theta}$. Obviously, $x_i \geq 1$ and

$$\begin{aligned}(2MN + (n-2)N^2)b\sum_{k=1}^{l-1}(x_k - 1) &= \frac{(2MN + (n-2)N^2)b\sum_{k=1}^{l-1}(s_k - s_l)}{\Phi_l - \theta} \\ &= \Phi_l - s_l.\end{aligned}$$

Thus for $1 \leq i \leq l-1$,

$$\begin{aligned}r_i(B) &\leq \frac{\Phi_l - s_l + (M^2 + (n-1)N^2 - 2MNb - (n-2)N^2b)\cdot\frac{s_i-s_l}{\Phi_l-\theta} + s_i}{1 + \frac{s_i-s_l}{\Phi_l-\theta}} \\ &= \Phi_l,\end{aligned}$$

and for $l \leq i \leq n$,

$$r_i(B) \leq \Phi_l - s_l + s_l = \Phi_l.$$

Now by Lemma 2.1, $\rho(A) = \sqrt{\rho(B)} \leq \sqrt{\max_{1\leq i\leq n} r_i(B)} \leq \sqrt{\Phi_l}$.

Suppose that $A$ is irreducible. Suppose that $\rho(A) = \sqrt{\Phi_l}$ for some $l$ with $2 \leq l \leq n$. Then $\rho(B) = \max_{1\leq i\leq n} r_i(B) = \Phi_l$.

If $A^2$ is irreducible, then so is $B$. By Lemma 2.1, $r_1(B) = \cdots = r_n(B) = \Phi_l$, and thus from the above arguments, (a) holds for $1 \leq i \leq l-1$, and (b) and (c) hold for $l \leq i \leq n$. Suppose that $s_1 > s_l$. Let $t$ be the smallest integer such that $s_t = s_l$, where $2 \leq t \leq l$. For $1 \leq k \leq t-1$, since $s_k > s_l$, we have $x_k > 1$. From (a) and (b), we have $a_{ii} = M$ and $a_{ij} = N$ for $1 \leq i,j \leq n$ with $j \neq i$, and thus $r_1 = \cdots = r_n = M + (n-1)N$, implying that $b = 1$, a contradiction. Then $s_1 = s_l$, and thus we have from (c) that $s_1 = \cdots = s_n$.

Suppose that $A^2$ is reducible. Then by Lemma 2.2, there is a permutation matrix $P$ such that $PAP^\top = \begin{pmatrix} 0_s & A_1 \\ A_2 & 0_{n-s}\end{pmatrix}$, where $A_2A_1$ and $A_1A_2$ are irreducible, and $\rho(A_2A_1) = \rho(A_1A_2) = \rho(A^2)$. Let $W = PUP^\top$. Obviously, $W$ is a diagonal matrix. We partition $W$ as $W = \begin{pmatrix} W_1 & 0_{s\times(n-s)} \\ 0_{(n-s)\times s} & W_2\end{pmatrix}$. Let $Y_1 = W_1^{-1}A_1A_2W_1$ and $Y_2 = W_2^{-1}A_2A_1W_2$. Obviously, $Y_1$ and $Y_2$ are irreducible. Then

$$PBP^\top = PU^{-1}P^\top\begin{pmatrix} A_1A_2 & 0_{s\times(n-s)} \\ 0_{(n-s)\times s} & A_2A_1\end{pmatrix}PUP^\top = \begin{pmatrix} Y_1 & 0_{s\times(n-s)} \\ 0_{(n-s)\times s} & Y_2\end{pmatrix}.$$

By Lemma 2.1,

$$\rho(Y_1) \leq \max_{1\leq i\leq s} r_i(Y_1) \leq \max_{1\leq i\leq n} r_i(PBP^\top) = \max_{1\leq i\leq n} r_i(B) = \Phi_l.$$

Note that $\rho(Y_1) = \rho(A_1A_2) = \rho(A^2) = \rho(B) = \Phi_l$. Thus $\rho(Y_1) = \max_{1\leq i\leq s} r_i(Y_1) = \Phi_l$. Since $Y_1$ is irreducible, $r_1(Y_1) = \cdots = r_s(Y_1) = \Phi_l$. Similarly, we have $r_1(Y_2) = \cdots = r_{n-s}(Y_2) = \Phi_l$. Thus $r_1(PBP^\top) = \cdots = r_n(PBP^\top) = \Phi_l$, i.e. $r_1(B) = \cdots = r_n(B) = \Phi_l$. By above argument, we have $s_1 = \cdots = s_n$.

Conversely, if $s_1 = \cdots = s_n$, then $\Phi_l = s_l$ for $1 \leq l \leq n$ and by Lemma 2.1, $\rho(B) = s_1$, and thus $\rho(A) = \sqrt{\rho(B)} = \sqrt{\Phi_l}$. □

In [7], the following upper bound for the spectral radius was given.

**Theorem 2.2:** *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $r_1, \dots, r_n$ and with average 2-row sums $m_1 \geq \cdots \geq m_n$. Let $M$ be the largest diagonal element, and $N$ the largest off-diagonal element of $A$. Let $b = \max\left\{\frac{r_i}{r_j} : 1 \leq i, j \leq n\right\}$. Suppose that $N > 0$. For $1 \leq l \leq n$, Let*

$$\Psi_l = \frac{m_l + M - Nb + \sqrt{(m_l - M + Nb)^2 + 4Nb\sum_{k=1}^{l-1}(m_k - m_l)}}{2}.$$

*Then $\rho(A) \leq \Psi_l$. Moreover,if $A$ is irreducible, then $\rho(A) = \Psi_l$ for some $l$ with $1 \leq l \leq n$ if and only if $m_1 = \cdots = m_n$, or for some $t$ with $2 \leq t \leq n$, $a_{ii} = M$ for $1 \leq i \leq t-1$, $a_{ik} = N$ and $\frac{r_k}{r_i} = b$ for $1 \leq i \leq n$ for $1 \leq k \leq t-1$ with $k \neq i$, and $m_t = \cdots = m_n$.*

Consider

$$A_1 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

In notation of Theorem 2.1, $s_1 = \frac{267}{7} \approx 38.1429$, $s_2 = 36$, $s_3 = \frac{104}{3} \approx 34.6667$, $s_4 = \frac{173}{5} = 34.6$, $M = 3$, $N = 2$, $b = \frac{7}{5}$, and $\theta = -7$, implying that $\sqrt{\Phi_1} \approx 6.176$, $\sqrt{\Phi_2} \approx 6.1117$, $\sqrt{\Phi_3} \approx 6.13846$ and $\sqrt{\Phi_4} \approx 6.1429$, and thus $\rho(A_1) \leq 6.1117$. In notation of Theorem 2.2, $m_1 = \frac{44}{7} \approx 6.2857$, $m_2 = 6$, $m_3 = \frac{35}{6} \approx 5.833$, $m_4 = \frac{29}{5} = 5.8$, $M = 3$, $N = 2$, and $b = \frac{7}{5}$, implying that $\Psi_1 \approx 6.2857$, $\Psi_2 \approx 6.1348$, $\Psi_3 \approx 6.1258$ and $\Psi_4 \approx 6.139$, and thus $\rho(A_1) \leq 6.1258$. The upper bound in Theorem 2.1 is smaller than the one in Theorem 2.2.

Consider

$$A_2 = \begin{pmatrix} 5 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 \end{pmatrix}.$$

In notation of Theorem 2.1, $s_1 = 178$, $s_2 = s_3 = s_4 = \frac{2305}{13} \approx 177.3077$, $M = 5$, $N = 4$, $b = \frac{14}{13}$, and $\theta = -\frac{59}{13}$, implying that $\sqrt{\Phi_1} \approx 13.3417$, $\sqrt{\Phi_2} = \sqrt{\Phi_3} = \sqrt{\Phi_4} \approx 13.3268$, and thus $\rho(A_2) \leq 13.3268$. In notation of Theorem 2.2, $m_1 = \frac{187}{14} \approx 13.3571$, $m_2 = m_3 = m_4 = \frac{173}{13} \approx 13.3077$, $M = 5$, $N = 4$ and $b = \frac{14}{13}$, implying that $\Psi_1 = \frac{187}{14}$, $\Psi_2 = \Psi_3 = \Psi_4 = 7 + \sqrt{40}$, and thus $\rho(A_2) \leq 7 + \sqrt{40}$. Note that $a_{11} = 5$, $a_{i1} = 4$ and $\frac{r_1}{r_i} = \frac{14}{13}$ for $2 \leq i \leq 4$, and $m_2 = m_3 = m_4$. Thus $\rho(A_2) = \Psi_2 = 7 + \sqrt{40} \approx 13.3246$. The upper bound in Theorem 2.2 is smaller than (but very close to) the one in Theorem 2.1.

The above examples show that in general the upper bounds in Theorems 2.1 and 2.2 are incomparable.

Next we give a sharp lower bound for the spectral radius of a nonnegative matrix.

**Theorem 2.3:** *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $r_1, \dots, r_n$ and with average 3-row sums $s_1 \geq \cdots \geq s_n$. Let $S$ be the smallest diagonal element, and $T$ the smallest off-diagonal element of $A$. Let $c = \min\left\{\frac{r_i}{r_j} : 1 \leq i, j \leq n\right\}$ and $\gamma = S^2 + (n-1)T^2 - 2STc - (n-2)T^2c$. Suppose that $s_n > \gamma$. Let*

$$\phi_n = \frac{s_n + \gamma + \sqrt{(s_n - \gamma)^2 + 4(2ST + (n-2)T^2)c\sum_{k=1}^{n-1}(s_k - s_n)}}{2}.$$

*Then $\rho(A) \geq \sqrt{\phi_n}$. Moreover, if $A$ is irreducible, then $\rho(A) = \sqrt{\phi_n}$ if and only if one of the following conditions holds:*

(i) *if $T = 0$, then $m_1(A) = \cdots = m_n(A)$ when $A^2$ is irreducible, and $PAP^\top = \begin{pmatrix} 0_r & A_1 \\ A_2 & 0_{n-r} \end{pmatrix}$ for some permutation matrix $P$ with $m_{\sigma(1)}(A) = \cdots = m_{\sigma(r)}(A)$ and $m_{\sigma(r+1)}(A) = \cdots = m_{\sigma(n)}(A)$ when $A^2$ is reducible, where $\sigma$ is a permutation on the set $\{1, \ldots, n\}$ which corresponds to the permutation matrix $P$;*

(ii) *if $T > 0$, then $s_1 = \cdots = s_n$.*

***Proof:*** If $T = 0$, then $\phi_n = s_n$, and thus the result follows immediately from Lemma 2.3.

Suppose in the following that $T > 0$.

Let $U = \mathrm{diag}(x_1 r_1, \ldots, x_{n-1} r_{n-1}, r_n)$, where $x_i \geq 1$ is a variable to be determined later for $1 \leq i \leq n-1$. Let $B = U^{-1}A^2U$. Obviously, $A^2$ and $B$ have the same eigenvalues. Then $\rho(A) = \sqrt{\rho(A^2)} = \sqrt{\rho(B)}$.

For $1 \leq i \leq n-1$, since $a_{ii} \geq S$, $\frac{r_k}{r_i} \geq c$ for $1 \leq k \leq n-1$ and $k \neq i$, and $a_{ij} \geq T$ for $1 \leq j \leq n$ and $j \neq i$, we have

$$\begin{aligned}
r_i(B) &= \frac{1}{x_i}\left[\sum_{\substack{1\leq k\leq n-1\\ k\neq i}}\left(a_{ii}a_{ik} + a_{ik}a_{kk} + \sum_{\substack{1\leq j\leq n\\ j\neq i,k}} a_{ij}a_{jk}\right)\frac{r_k}{r_i}(x_k - 1)\right.\\
&\qquad \left. + \left(a_{ii}a_{ii} + \sum_{\substack{1\leq j\leq n\\ j\neq i}} a_{ij}a_{ji}\right)(x_i - 1) + s_i\right]\\
&\geq \frac{1}{x_i}\left(\sum_{\substack{1\leq k\leq n-1\\ k\neq i}}\left(2ST + (n-2)T^2\right)c(x_k - 1) + (S^2 + (n-1)T^2)(x_i - 1) + s_i\right)\\
&= \frac{1}{x_i}\left(\left(2ST + (n-2)T^2\right)c\sum_{k=1}^{n-1}(x_k - 1)\right.\\
&\qquad \left. + (S^2 + (n-1)T^2 - 2STc - (n-2)T^2c)(x_i - 1) + s_i\right)
\end{aligned}$$

with equality when $x_k > 1$ for $1 \leq k \leq n-1$ only if (a) holds: (a) $a_{kk} = S$, $a_{jk} = T$ for $1 \leq j \leq n$ with $j \neq k$.

Similarly, we have

$$\begin{aligned}
r_n(B) &\geq \sum_{k=1}^{n-1}\left(2ST + (n-2)T^2\right)c(x_k - 1) + s_n\\
&= \left(2ST + (n-2)T^2\right)c\sum_{k=1}^{n-1}(x_k - 1) + s_n
\end{aligned}$$

with equality $x_k > 1$ for $1 \le k \le n-1$ only if (b) holds: (b) $a_{nn} = a_{kk} = S$ and $a_{nj} = T$ for $1 \le j \le n-1$.

From the expression of $\phi_n$, we have $\phi_n \ge \frac{s_n+\gamma+|s_n-\gamma|}{2} = s_n > \gamma$. For $1 \le i \le n-1$, let $x_i = 1 + \frac{s_i - s_n}{\phi_n - \gamma}$. Obviously, $x_i \ge 1$ and

$$\begin{aligned}(2ST + (n-2)T^2)c\sum_{k=1}^{n-1}(x_k - 1) &= \frac{(2ST + (n-2)T^2)c\sum_{k=1}^{n-1}(s_k - s_n)}{\phi_n - \gamma} \\ &= \phi_n - s_n.\end{aligned}$$

Thus for $1 \le i \le n-1$,

$$\begin{aligned}r_i(B) &\ge \frac{\phi_n - s_n + (S^2 + (n-1)T^2 - 2STc - (n-2)T^2c)\cdot\frac{s_i - s_n}{\phi_n - \gamma} + s_i}{1 + \frac{s_i - s_n}{\phi_n - \gamma}} \\ &= \phi_n,\end{aligned}$$

and

$$r_n(B) \ge \phi_n - s_n + s_n = \phi_n.$$

Hence, by Lemma 2.1, $\rho(A) \ge \sqrt{\rho(B)} \ge \sqrt{\min_{1\le i\le n} r_i(B)} \ge \sqrt{\phi_n}$.

Suppose that $A$ is irreducible. If $\rho(A) = \sqrt{\phi_n}$, then $\rho(B) = \min_{1\le i\le n} r_i(B) = \phi_n$, and thus by similar arguments as in the proof of Theorem 2.1, we have $s_1 = \cdots = s_n$.

Conversely, if $s_1 = \cdots = s_n$, then $\phi_n = s_n$ and by Lemma 2.1, $\rho(B) = s_n = \phi_n$, and thus $\rho(A) = \sqrt{\rho(B)} = \sqrt{\phi_n}$. □

In [7], the following lower bound for the spectral radius was given.

**Theorem 2.4:** *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $r_1, \ldots, r_n$ and with average 2-row sums $m_1 \ge \cdots \ge m_n$. Let $S$ be the smallest diagonal element, and $T$ the smallest off-diagonal element of $A$. Let $c = \min\left\{\frac{r_j}{r_i} : 1 \le i, j \le n\right\}$. Let*

$$\psi_n = \frac{m_n + S - Tc + \sqrt{(m_n - S + Tc)^2 + 4Tc\sum_{k=1}^{n-1}(m_k - m_n)}}{2}.$$

*Then $\rho(A) \ge \psi_n$. Moreover, if $A$ is irreducible, then $\rho(A) = \psi_n$ if and only if $m_1 = \cdots = m_n$, or $T > 0$ and for some $t$ with $2 \le t \le n$, $a_{ii} = S$ for $1 \le i \le t-1$, $a_{ik} = T$ and $\frac{r_k}{r_i} = c$ for $1 \le i \le n$ for $1 \le k \le t-1$ with $k \ne i$, and $m_t = \cdots = m_n$.*

Consider

$$A_3 = \begin{pmatrix} 4 & 2 & 1 & 1 \\ 2 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$

In notation of Theorem 2.3, $s_1 = \frac{257}{3} \approx 85.6667$, $s_2 = s_3 = \frac{829}{10} = 82.9$, $s_4 = 79$, $S = 1$, $T = 1$, $c = \frac{4}{5}$, and $\gamma = \frac{4}{5}$, implying that $\sqrt{\phi_4} \approx 8.89$, and thus $\rho(A_3) \ge 8.89$. Obviously, $A_3$ is permutation similar to

$$A_3' = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 3 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 2 & 1 & 1 & 4 \end{pmatrix},$$

and then $\rho(A_3') = \rho(A_3)$. In notation of Theorem 2.4, $m_1 = \frac{85}{9} \approx 9.4444$, $m_2 = m_3 = \frac{91}{10} = 9.1$, $m_4 = \frac{35}{4} = 8.75$, $S = 1$, $T = 1$, and $c = \frac{4}{5}$, implying that $\psi_4 \approx 8.8785$, and thus $\rho(A_3') \geq 8.8785$. The lower bound in Theorem 2.3 is larger than the one in Theorem 2.4.

Consider

$$A_4 = \begin{pmatrix} 1 & 5 & 3 & 3 \\ 3 & 4 & 4 & 4 \\ 3 & 4 & 4 & 4 \\ 3 & 4 & 4 & 4 \end{pmatrix}.$$

In notation of Theorem 2.3, $s_1 = \frac{851}{4} = 212.75$, $s_2 = s_3 = s_4 = \frac{1041}{5} = 208.2$, $S = 1$, $T = 3$, $c = \frac{4}{5}$, and $\gamma = \frac{44}{5}$, implying that $\sqrt{\phi_4} \approx 14.4434$, and thus $\rho(A_3) \geq 14.4443$. In notation of Theorem 2.4, $m_1 = \frac{59}{4} = 14.75$, $m_2 = m_3 = m_4 = \frac{72}{5} = 14.4$, $S = 1$, $T = 3$, and $c = \frac{4}{5}$, implying that $\psi_4 = \frac{13+\sqrt{253}}{2}$, and thus $\rho(A_4) \geq \frac{13+\sqrt{253}}{2}$. Note that $a_{11} = 1$, $a_{i1} = 3$ and $\frac{r_1}{r_i} = \frac{4}{5}$ for $2 \leq i \leq 4$, and $m_2 = m_3 = m_4$. We have $\rho(A_4) = \frac{13+\sqrt{253}}{2} \approx 14.45299$. The lower bound in Theorem 2.4 is larger than (but very close to) the one in Theorem 2.3.

The above examples show that in general the lower bounds in Theorems 2.3 and 2.4 are incomparable.

## 3. Applications

In this section, we consider the applications of Theorems 2.1 and 2.3 to some matrices associated to digraphs and graphs.

First we consider digraphs.

Let $\overrightarrow{G}$ be an $n$-vertex digraph with $\delta^+ > 0$, where $V(\overrightarrow{G}) = \{v_1, \ldots, v_n\}$ and $\delta^+$ is the minimum out-degree of $\overrightarrow{G}$. Let $\Delta^+$ be the maximum out-degree of $\overrightarrow{G}$. For $1 \leq i \leq n$, $m_i(A(\overrightarrow{G})) = \frac{\sum_{(v_i,v_j)\in E(\overrightarrow{G})} d_j^+}{d_i^+}$, which is known as the average 2-out-degree of vertex $v_i$ in $\overrightarrow{G}$, and $s_i(A(\overrightarrow{G})) = \frac{\sum_{(v_i,v_j)\in E(\overrightarrow{G})} \sum_{(v_j,v_k)\in E(\overrightarrow{G})} d_k^+}{d_i^+}$, which we call the average 3-out-degree of vertex $v_i$ in $\overrightarrow{G}$.

A digraph $\overrightarrow{G}$ is bipartite if $V(\overrightarrow{G}) = X \cup Y$, $X \cap Y = \emptyset$, and the arc set is a subset of $(X \times Y) \cup (Y \times X)$. Here $X$ and $Y$ are the partite sets.

**Corollary 3.1:** *Let $\overrightarrow{G}$ be a digraph on $n$ vertices with minimum out-degree $\delta^+ > 0$ and average 3-out-degrees $s_1 \geq \cdots \geq s_n$. Let $\theta = n - 1 - \frac{(n-2)\Delta^+}{\delta^+}$. Then for $1 \leq l \leq n$,*

$$\rho(A(\overrightarrow{G})) \leq \sqrt{\frac{s_l + \theta + \sqrt{(s_l - \theta)^2 + \frac{4(n-2)\Delta^+}{\delta^+} \sum_{k=1}^{l-1} (s_k - s_l)}}{2}}.$$

*Moreover, if $\overrightarrow{G}$ is strongly connected, equality holds for some $1 \leq l \leq n$ if and only if $\overrightarrow{G}$ is a non-bipartite digraph with equal average 2-out-degrees or $\overrightarrow{G}$ is a bipartite digraph in* *which vertices in the same partite set have equal average 2-out-degrees when $l = 1$, and $s_1 = \cdots = s_n$ when $2 \le l \le n$.*

***Proof:*** In the notation of Theorem 2.1, $M = 0$, $N = 1$ and $b = \frac{\Delta^+}{\delta^+}$. If $b = 1$, then $s_1 = \cdots = s_n$, and thus $s_1 \ge 1 = \theta$. If $b > 1$, then $b \ge \frac{n-1}{n-2}$ and $\theta \le 0$, from which we have $s_1 > \theta$. If $\overrightarrow{G}$ is strongly connected, then $A(\overrightarrow{G})$ is irreducible, and by Lemma 2.2, $(A(\overrightarrow{G}))^2$ is irreducible if and only if $\overrightarrow{G}$ is not bipartite. Thus the result follows from Theorem 2.1. □

For $1 \le i \le n$, $m_i(Q(\overrightarrow{G})) = d_i^+ + \frac{1}{d_i^+}\sum_{(v_i,v_j)\in E(\overrightarrow{G})} d_j^+$, which is known as the signless Laplacian average 2-out-degree of vertex $v_i$ in G, and

$$s_i(Q(\overrightarrow{G})) = {d_i^+}^2 + \sum_{(v_i,v_j)\in E(\overrightarrow{G})} d_j^+ + \frac{1}{d_i^+}\sum_{(v_i,v_j)\in E(\overrightarrow{G})}\left({d_j^+}^2 + \sum_{(v_j,v_k)\in E(\overrightarrow{G})} d_k^+\right),$$

which we call the signless Laplacian average 3-out-degree of vertex $v_i$ in $\overrightarrow{G}$.

**Corollary 3.2:** *Let $\overrightarrow{G}$ be a digraph on $n$ vertices with minimum out-degree $\delta^+ > 0$ and signless Laplacian average 3-out-degrees $s_1 \ge \cdots \ge s_n$. Let $\theta = (\Delta^+)^2 + (n-1) - \frac{2(\Delta^+)^2}{\delta^+} - \frac{(n-2)\Delta^+}{\delta^+}$. Then for $1 \le l \le n$,*

$$\rho(Q(\overrightarrow{G})) \le \sqrt{\frac{s_l + \theta + \sqrt{(s_l-\theta)^2 + 4(2\Delta^+ + n - 2)\frac{\Delta^+}{\delta^+}\sum_{k=1}^{l-1}(s_k - s_l)}}{2}}.$$

*Moreover, if $\overrightarrow{G}$ is strongly connected, equality holds for some $1 \le l \le n$ if and only if $\overrightarrow{G}$ has equal signless Laplacian average 2-out-degrees when $l = 1$, and $s_1 = \cdots = s_n$ when $2 \le l \le n$.*

***Proof:*** In the notation of Theorem 2.1, $M = \Delta^+$, $N = 1$ and $b = \frac{\Delta^+}{\delta^+}$. If $b = 1$, then $s_1 = \cdots = s_n$ and $s_1 \ge (\Delta^+ - 1)^2 = \theta$. If $b > 1$, then $b \ge \frac{n-1}{n-2}$, and thus $\theta < 0$, from which we have $s_1 > \theta$. If $\overrightarrow{G}$ is strongly connected, then $Q(\overrightarrow{G})$ and $(Q(\overrightarrow{G}))^2$ are irreducible. Thus the result follows from Theorem 2.1, □

Next we consider graphs.

Let $G$ be an $n$-vertex graph without isolated vertices, where $V(G) = \{v_1, \ldots, v_n\}$. Let $\Delta$ and $\delta$ be the maximum and minimum degree of $G$, respectively. For $1 \le i \le n$, $m_i(A(G)) = \frac{\sum_{v_iv_j\in E(G)} d_j}{d_i}$, which is known as the average 2-degree of vertex $v_i$ in $G$, and $s_i(A(G)) = \frac{\sum_{v_iv_j\in E(G)}\sum_{v_jv_k\in E(G)} d_k}{d_i}$, which we call the average 3-degree of vertex $v_i$ in $G$.

By Corollary 3.1, we have

**Corollary 3.3:** *Let $G$ be a graph on $n$ vertices without isolated vertices with average 3-degrees $s_1 \ge \cdots \ge s_n$. Let $\theta = n - 1 - (n-2)\frac{\Delta}{\delta}$. Then for $1 \le l \le n$,*

$$\rho(A(G)) \le \sqrt{\frac{s_l + \theta + \sqrt{(s_l-\theta)^2 + \frac{4(n-2)\Delta}{\delta}\sum_{k=1}^{l-1}(s_k - s_l)}}{2}}.$$

*Moreover, if G is connected, then equality holds for some* $1 \le l \le n$ *if and only if G is a non-bipartite graph with equal average 2-degrees or G is a bipartite graph in which vertices in the same partite set have equal average 2-degrees when* $l = 1$*, and* $s_1 = \cdots = s_n$ *when* $2 \le l \le n$.

Let $H$ be a graph obtained by attaching one pendant vertex to each pendant vertex of a 4-vertex star. It is easy seen that each vertex of $H$ has the same average 3-degree 4. By Corollary 3.3, $\rho(H) = 2$.

For $1 \le i \le n$, $m_i(Q(G)) = d_i + \frac{1}{d_i}\sum_{v_iv_j\in E(G)} d_j$, which is known as the signless Laplacian average 2-degree of vertex $v_i$ in $G$, and

$$s_i(Q(G)) = d_i^2 + \sum_{v_iv_j\in E(G)} d_j + \frac{\sum_{v_iv_j\in E(G)}\left(d_j^2 + \sum_{v_jv_k\in E(G)} d_k\right)}{d_i},$$

which we call the signless Laplacian average 3-degree of vertex $v_i$ in $G$.

By Corollary 3.2, we have

**Corollary 3.4:** *Let G be a graph on n vertices without isolated vertices with signless Laplacian average 3-degrees* $s_1 \ge \cdots \ge s_n$. *Let* $\theta = \Delta^2 + (n-1) - \frac{2\Delta^2}{\delta} - \frac{\Delta(n-2)}{\delta}$. *Then for* $1 \le l \le n$,

$$\rho(Q(G)) \le \sqrt{\frac{s_l + \theta + \sqrt{(s_l - \theta)^2 + 4(2\Delta + n - 2)\frac{\Delta}{\delta}\sum_{k=1}^{l-1}(s_k - s_l)}}{2}}.$$

*Moreover, if G is connected, the equality holds for some* $1 \le l \le n$ *if and only if G has equal signless Laplacian average 2-degrees when* $l = 1$*, and* $s_1 = \cdots = s_n$ *when* $2 \le l \le n$.

The 5-vertex star $S_5$ is an irregular graph with the same signless Laplacian average 3-degree 25 for each vertex. By Corollary 3.4, $\rho(S_5) = 5$.

## 4. Remarks

In the literature, upper and lower bounds have been obtained for the spectral radius of a nonnegative matrix using row sums and average 2-row sums. For a nonnegative matrix with positive row sums, maximum and minimum average 3-row sums have been used, respectively, to give upper and lower bounds for the spectral radius in [8]. In this paper, we give sharp upper and lower bounds for the spectral radius using average 3-row sums, and characterize the equality cases if the matrix is irreducible. Even in form, these bounds are different from the ones using the average 2-row sums.[7] Finally, these bounds are applied to the adjacency matrix and the signless Laplacian matrix of a digraph or a graph.

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## References


[1] Horn RA, Johnson CR. Matrix analysis. Cambridge: Cambridge University Press; 1985.
[2] Cohen JE. Derivatives of the spectral radius as a function of non-negative matrix elements. Math Proc Cambridge Philos Soc. 1978;83:183–190.
[3] Duan X, Zhou B. Sharp bounds on the spectral radius of a nonnegative matrix. Linear Algebra Appl. 2013;439:2961–2970.
[4] Melman A. Upper and lower bounds for the Perron root of a nonnegative matrix. Linear Multilinear Algebra. 2013;61:171–181.
[5] Minc H. Nonnegative matrices. New York (NY): Wiley; 1988.
[6] Tasci D, Kirkland S. A sequence of upper bounds for the Perron root of a nonnegative matrix. Linear Algebra Appl. 1998;273:23–28.
[7] Xing R, Zhou B. Sharp bounds for the spectral radius of nonnegative matrices. Linear Algebra Appl. 2014;449:194–209.
[8] Zhang XD, Li JS. Spectral radius of non-negative matrices and digraphs. Acta Math Sin (Engl Ser). 2002;18:293–300.
[9] Walters P. An introduction to ergodic theory. New York (NY): Springer-Verlag; 1982.
[10] Brualdi RA. Spectra of digraphs. Linear Algebra Appl. 2010;432:2181–2213.
[11] Jin YL, Zhang XD. On the spectral radius of simple digraphs with prescribed number of arcs. Discrete Math. 2015;338:1555–1564.
[12] Xu GH, Xu CQ. Sharp bounds for the spectral radius of digraphs. Linear Algebra Appl. 2009;430:1607–1612.
[13] Bozkurt SB, Bozkurt D. On the signless Laplacian spectral radius of digraphs. Ars Combin. 2013;108:193–200.
[14] Cvetković D, Doob M, Sachs H. Spectra of graphs. New York (NY): Academic Press; 1980.
[15] Cvetković D, Rowlinson P. The largest eigenvalue of a graph: a survey. Linear Multilinear Algebra. 1990;28:3–33.
[16] Cvetković D, Rowlinson P, Simić SK. Signless Laplacians of finite graphs. Linear Algebra Appl. 2007;423:155–171.
[17] Hansen P, Lucas C. Bounds and conjectures for the signless Laplacian index of graphs. Linear Algebra Appl. 2010;432:3319–3336.