Research Article

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Bounds for the spectral radius of nonnegative matrices and generalized Fibonacci matrices

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Abstract: In this article, we determine upper and lower bounds for the spectral radius of nonnegative matrices. Introducing the notion of average 4-row sum of a nonnegative matrix, we extend various existing formulas for spectral radius bounds. We also refer to their equality cases if the matrix is irreducible, and we present numerical examples to make comparisons among them. Finally, we provide an application to special matrices such as the generalized Fibonacci matrices, which are widely used in applied mathematics and computer science problems.

Keywords: nonnegative matrix, spectral radius, 4-row sum, average 4-row sum, generalized Fibonacci matrix

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1 Introduction

The problem of bounding the largest in magnitude eigenvalue, the spectral radius, of a nonnegative matrix has been encountered in several branches of applied mathematics and computer science such as graph theory, probability, cryptography, and even biology with applications to various stochastic and dynamic processes (e.g., rumor, disease, information spread). For instance, the epidemic threshold in a real-world network has been shown to be inversely proportional to the spectral radius of the adjacency matrix, thus the principal eigenvalue serves as a measure to quantify the robustness against virus propagation [1].

In this work, we develop lower and upper bounds for the spectral radius of a nonnegative matrix generalizing upon existing formulations [2–7]. Our framework introduces the 4-row sums and the average 4-row sums of the matrix to accommodate alternative bounded regions for the spectral radius. In addition, equality cases of the bounds are characterized for nonnegative and irreducible matrices. We elaborate on the specific results after providing the necessary notation next.

Let $\mathcal{M}_{m,n}(\mathbb{R})$ be the algebra of $m \times n$ real matrices, where the case $m = n$ is specified by $\mathcal{M}_n(\mathbb{R})$. We recall that $A = (a_{ij})_{i,j=1}^n \in \mathcal{M}_n(\mathbb{R})$ is *nonnegative* when each $a_{ij} \geq 0$, denoted by $A \geq 0$. Similarly, A is *positive* whenever each $a_{ij} > 0$, denoted by $A > 0$. The matrix A is *irreducible* if and only if $(I + A)^{n-1} > 0$. We define the *spectral radius* of A by

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the set of eigenvalues of A [8].

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For $1 \le i, k \le n$, $r_i(A) = \sum_{j=1}^{n} a_{ij}$ is known as the ith row sum of A and for $r_i(A) > 0$ the quantities

$$M_i(A) = \sum_{j=1}^{n} a_{ij} r_j(A) \quad \text{and} \quad m_i(A) = \frac{M_i(A)}{r_i(A)} = \sum_{j=1}^{n} a_{ij} \frac{r_j(A)}{r_i(A)} \tag{1.1}$$

are called the ith 2-row sum and the ith average 2-row sum of A, respectively [6,7]. The quantities

$$\begin{aligned} S_i(A) &= \sum_{j=1}^{n} a_{ij} M_j(A) = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij} a_{jk} r_k(A) \\ s_i(A) &= \frac{S_i(A)}{r_i(A)} = \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij} M_j(A) = \frac{1}{r_i(A)} \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij} a_{jk} r_k(A) \end{aligned} \tag{1.2}$$

are called the ith 3-row sum and the ith average 3-row sum of A, respectively [6].

Motivated by the expressions of $M_i(A)$, $m_i(A)$ in (1.1) and $S_i(A)$, $s_i(A)$ in (1.2), we introduce the ith *4-row sum* of A defined by

$$W_i(A) = \sum_{j=1}^{n} a_{ij} S_j(A) = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n} a_{ij} a_{jk} a_{kp} r_p(A),$$

and the ith *average 4-row* sum of A defined by the ratio

$$w_i(A) = \frac{W_i(A)}{r_i(A)} = \frac{1}{r_i(A)} \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n} a_{ij} a_{jk} a_{kp} r_p(A). \tag{1.3}$$

Then, we notice that

$$w_i(A) = \sum_{j=1}^{n} a_{ij} \frac{S_j(A)}{r_i(A)} = \sum_{j=1}^{n} a_{ij}^{(2)} \frac{M_j(A)}{r_i(A)} = \sum_{j=1}^{n} a_{ij}^{(3)} \frac{r_j(A)}{r_i(A)} = \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij}^{(4)}, \tag{1.4}$$

where $a_{ij}^{(k)}$ denotes the (i,j)th element of the power of the matrix A^k for $k = 2, 3, 4$. Moreover, the following extreme entries of the matrix A will be used: its largest diagonal and off-diagonal elements,

$$\mu = \max_{1 \le i \le n} \{a_{ii}\} \quad \text{and} \quad \nu = \max_{\substack{1 \le i,j \le n \\ i \ne j}} \{a_{ij}\}, \tag{1.5}$$

respectively, as well as its smallest diagonal and off-diagonal elements,

$$\zeta = \min_{1 \le i \le n} \{a_{ii}\} \quad \text{and} \quad \eta = \min_{\substack{1 \le i,j \le n \\ i \ne j}} \{a_{ij}\}, \tag{1.6}$$

respectively.

This article is organized as follows. In Section 2, we establish upper bounds for the spectral radius of a nonnegative matrix by using its average 4-row sums. Likewise, Section 3 presents a lower bound. The equality cases are discussed in both sections whenever the matrix is irreducible. Numerical examples are also displayed to compare among all existing formulas in [2,5–7]. They illustrate that the proposed bounds are shown to be tighter in some cases. Finally, in Section 4, we apply the previous bounds to the generalized k-Fibonacci matrix.

2 Upper bound for the spectral radius of nonnegative matrices

In this section, we investigate some upper bounds for the spectral radius $\rho(A)$ of a nonnegative matrix A, generalizing upon established results [5–7]. Adopting the techniques used therein, we obtain new expressions for the upper bound of $\rho(A)$. The following arguments will be used in our results.

Proposition 2.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ and $A^3 = (a_{ij}^{(3)})_{i,j=1}^n \in \mathcal{M}_n(\mathbb{R})$. Let μ, ν be the largest diagonal and off-diagonal elements of A and ζ, η be the smallest diagonal and off-diagonal elements of A, respectively. Consider*

$$\alpha_1 = (n-1)(n-2)\nu^3 + 3(n-1)\mu\nu^2 + \mu^3, \quad \alpha_2 = \alpha_1 + (\nu - \mu)^3 \tag{2.1}$$

and

$$\beta_1 = (n-1)(n-2)\eta^3 + 3(n-1)\zeta\eta^2 + \zeta^3, \quad \beta_2 = \beta_1 + (\eta - \zeta)^3. \tag{2.2}$$

Then, for $1 \leq i \leq n$ holds

$$\beta_1 \leq a_{ii}^{(3)} \leq \alpha_1, \tag{2.3}$$

and for $i \neq j$,

$$\beta_2 \leq a_{ij}^{(3)} \leq \alpha_2. \tag{2.4}$$

Right-hand side equalities hold only if $a_{ii} = \mu$ and $a_{ij} = \nu$, for all $i, j = 1, \ldots, n$, with $j \neq i$, while equalities at the left side hold only if $a_{ii} = \zeta$ and $a_{ij} = \eta$, for all $i, j = 1, \ldots, n$, with $j \neq i$.

Proof. Obviously, for $1 \leq i \leq n$, the diagonal elements of A^2 can be bounded from above

$$a_{ii}^{(2)} = \sum_{k=1}^{n} a_{ik}a_{ki} = a_{ii}^2 + \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{ik}a_{ki} \leq \mu^2 + (n-1)\nu^2, \tag{2.5}$$

where equality holds only if $a_{ii} = \mu$ and $a_{ik} = \nu$ for $i, k = 1, \ldots, n$ with $k \neq i$. Moreover, for $i, j = 1, \ldots, n$ with $j \neq i$ the off-diagonal elements of A^2 are bounded from above

$$a_{ij}^{(2)} = a_{ii}a_{ij} + a_{ij}a_{jj} + \sum_{\substack{k=1 \\ k \neq i,j}}^{n} a_{ik}a_{kj} \leq 2\mu\nu + (n-2)\nu^2, \tag{2.6}$$

where equality holds when $a_{ii} = a_{jj} = \mu$ and $a_{ij} = \nu$ for $i, j = 1, \ldots, n$ with $j \neq i$.

Likewise, if we use the elements ζ, η from (1.6), then the diagonal and off-diagonal elements of A^2 are bounded from below

$$a_{ii}^{(2)} \geq \zeta^2 + (n-1)\eta^2, \quad a_{ij}^{(2)} \geq 2\zeta\eta + (n-2)\eta^2, \tag{2.7}$$

with equalities when $a_{ii} = a_{jj} = \zeta$ and $a_{ij} = \eta$ for $i, j = 1, \ldots, n$ with $j \neq i$.

Substituting (2.5) and (2.6) into the diagonal elements of A^3, the following inequality arises

$$\begin{aligned} a_{ii}^{(3)} = a_{ii}^{(2)}a_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{ik}^{(2)}a_{ki} &\leq [\mu^2 + (n-1)\nu^2]\mu + (n-1)[2\mu\nu + (n-2)\nu^2]\nu \\ &= (n^2 - 3n + 2)\nu^3 + 3(n-1)\mu\nu^2 + \mu^3 = \alpha_1. \end{aligned}$$

It is apparent that equality holds when $a_{ii} = \mu$ and $a_{ik} = \nu$ for $i, k = 1, \ldots, n$ with $k \neq i$. Analogously, substituting both (2.5) and (2.6) into the off-diagonal elements of A^3, the following inequality arises

$$\begin{aligned} a_{ij}^{(3)} &= a_{ii}^{(2)}a_{ij} + a_{ij}^{(2)}a_{jj} + \sum_{\substack{k=1 \\ k \neq i,j}}^{n} a_{ik}^{(2)}a_{kj} \\ &\leq [\mu^2 + (n-1)\nu^2]\nu + [2\mu\nu + (n-2)\nu^2]\mu + (n-2)[2\mu\nu + (n-2)\nu^2]\nu \\ &= (n^2 - 3n + 3)\nu^3 + 3(n-2)\mu\nu^2 + 3\mu^2\nu = \alpha_1 + (\nu - \mu)^3 = \alpha_2. \end{aligned}$$

Equality holds when $a_{ii} = \mu$ and $a_{ik} = \nu$ for $i, k = 1, \ldots, n$ with $k \neq i$.

Similar arguments apply for the lower bounds of the diagonal and off-diagonal elements of A^3 with respect to the quantities ζ, η. In particular, by (2.7),

$$\begin{aligned} a_{ii}^{(3)} &\geq [\zeta^2 + (n-1)\eta^2]\zeta + (n-1)[2\zeta\eta + (n-2)\eta^2]\eta \\ &= (n^2 - 3n + 2)\eta^3 + 3(n-1)\zeta\eta^2 + \zeta^3 = \beta_1, \end{aligned}$$

$$\begin{aligned} a_{ij}^{(3)} &\geq [\zeta^2 + (n-1)\eta^2]\eta + [2\zeta\eta + (n-2)\eta^2]\zeta + (n-2)[2\zeta\eta + (n-2)\eta^2]\eta \\ &= (n^2 - 3n + 3)\eta^3 + 3(n-2)\zeta\eta^2 + 3\zeta^2\eta = \beta_1 + (\eta - \zeta)^3 = \beta_2, \end{aligned}$$

which reduce to equalities when $a_{ii} = \zeta$ and $a_{ij} = \eta$ for $i, j = 1, \ldots, n$ with $j \neq i$. ◻

The next lemma demonstrates well-established bounds for $\rho(A)$, and since it will be used in the proof of our results, it is stated herein for the sake of completeness.

Lemma 2.2. [8,9] *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with ith row sum $r_i(A)$, $i = 1, \ldots, n$ and let μ be the largest diagonal element of A. Then, $\rho(A) \geq \mu$ and*

$$\min_{1\leq i\leq n}\{r_i(A)\} \leq \rho(A) \leq \max_{1\leq i\leq n}\{r_i(A)\}.$$

If A is also irreducible, then either equality holds if and only if $r_1(A) = \cdots = r_n(A)$.

The next lemma is derived as an immediate consequence of Corollary 3.1 and Theorem 3.3. in ref. [4].

Lemma 2.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_i(A) > 0$, $1 \leq i \leq n$. Then,*

$$\min_{1\leq i\leq n}\{\sqrt[3]{w_i(A)}\} \leq \rho(A) \leq \max_{1\leq i\leq n}\{\sqrt[3]{w_i(A)}\}.$$

Let also A be irreducible.

(i) *If A^3 is irreducible, then either equality holds if and only if $m_1(A) = \cdots = m_n(A)$.*

(ii) *If A^3 is reducible, then either equality holds if and only if there is a permutation matrix P such that $PAP^T = \begin{pmatrix} 0_{n_1} & A_1 & 0 \\ 0 & 0_{n_2} & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$, and $m_{j_1}(A) = \cdots = m_{j_{n_1}}(A)$, $m_{j_{n_1+1}}(A) = \cdots = m_{j_{n_1+n_2}}(A)$, $m_{j_{n_1+n_2+1}}(A) = \cdots = m_{j_n}(A)$, where the sets $\{j_1, \ldots, j_{n_1}\}$, $\{j_{n_1+1}, \ldots, j_{n_1+n_2}\}$, $\{j_{n_1+n_2+1}, \ldots, j_n\}$ form a partition of $\{1, \ldots, n\}$ corresponding to the permutation matrix P.*

Next we state and prove the main result in this section.

Theorem 2.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_i(A) > 0$ for all $i = 1, \ldots, n$, and average 4-row sums $w_1(A) \geq w_2(A) \geq \cdots \geq w_n(A)$. Let μ, ν be the largest diagonal and off-diagonal elements of A, respectively, with $\nu > 0$. Denote by $b = \max\left\{\frac{r_j(A)}{r_i(A)} : 1 \leq i, j \leq n\right\}$, and $\gamma = \alpha_1 - b\alpha_2$ with α_1, α_2 given by (2.1). Assuming $w_1(A) \geq \gamma$, when $b = 1$, and $w_1(A) > \gamma$, when $b > 1$, let*

$$Z_\ell = \frac{1}{2}(w_\ell(A) + \gamma + \sqrt{\Delta_\ell}), \quad \ell = 1, \ldots, n, \tag{2.8}$$

where

$$\Delta_\ell = (w_\ell(A) - \gamma)^2 + 4b\alpha_2 \sum_{j=1}^{\ell-1}(w_j(A) - w_\ell(A)). \tag{2.9}$$

Then,

$$\rho(A) \leq \min\{\sqrt[3]{Z_\ell} : 1 \leq \ell \leq n\}. \tag{2.10}$$

Proof. To simplify the exposition of our calculations, we let $r_i = r_i(A)$ and $w_i = w_i(A)$ for $1 \le i \le n$. Consider $\ell = 1$ in (2.8); then from the assumption $w_1 > \gamma$ and (2.9) arises

$$Z_1 = \frac{1}{2}(w_1 + \gamma + |w_1 - \gamma|) = \frac{1}{2}(w_1 + \gamma + w_1 - \gamma) = w_1,$$

and the result follows immediately from Lemma 2.3.

Consider $2 \le \ell \le n$. If $b = 1$, then $r_1 = \cdots = r_n$ clearly implies that $w_1 = \cdots = w_n$. Hence, $Z_\ell = w_\ell = w_1$ for any ℓ and by Lemma 2.3, $\rho(A) = \sqrt[3]{w_1} = \sqrt[3]{Z_\ell}$. If $b > 1$, then let $U = \operatorname{diag}(r_1x_1, \ldots, r_{\ell-1}x_{\ell-1}, r_\ell, \ldots, r_n)$ be an $n \times n$ diagonal matrix, where $x_j \ge 1$ is a variable to be determined later for $1 \le j \le \ell - 1$ and let $B = U^{-1}A^3U$. Due to similarity, A^3 and B have the same eigenvalues; hence, $\rho(A) = \sqrt[3]{\rho(A^3)} = \sqrt[3]{\rho(B)}$.

For $1 \le i \le \ell - 1$, we derive

$$\begin{aligned}
r_i(B) &= \frac{1}{x_i}\left(\sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}x_j + \sum_{j=\ell}^{n} a_{ij}^{(3)}\frac{r_j}{r_i}\right)\\
&= \frac{1}{x_i}\left(\sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}x_j + a_{ii}^{(3)}x_i + \sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} - \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}\right)\\
&= \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} + \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}x_j - \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i} + a_{ii}^{(3)}x_i - a_{ii}^{(3)}\right)\\
&= \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} + \sum_{\substack{j=1\\ j\ne i}}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}(x_j - 1) + a_{ii}^{(3)}(x_i - 1)\right).
\end{aligned} \tag{2.11}$$

By (1.4), the right part of the inequalities in (2.3)–(2.4), $\frac{r_j}{r_i} \le b$ for $1 \le j \le \ell - 1$ and $j \ne i$ and the definition $\gamma = \alpha_1 - b\alpha_2$, the equality in (2.11) is formulated as follows:

$$r_i(B) \le \frac{1}{x_i}\left(w_i + b\alpha_2 \sum_{\substack{j=1\\ j\ne i}}^{\ell-1}(x_j - 1) + \alpha_1(x_i - 1)\right) \tag{2.12}$$

$$= \frac{1}{x_i}\left(w_i + b\alpha_2 \sum_{j=1}^{\ell-1}(x_j - 1) + \gamma(x_i - 1)\right). \tag{2.13}$$

Analogously for $\ell \le i \le n$, we use the equality in (1.4), the inequality in (2.4), as well as the assumptions $w_i \le w_\ell$, and $\frac{r_j}{r_i} \le b$ to derive

$$\begin{aligned}
r_i(B) &= \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}x_j + \sum_{j=\ell}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} = \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}x_j + \sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} - \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}\\
&= \sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} + \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}(x_j - 1) = w_i + \sum_{j=1}^{\ell-1} a_{ij}^{(3)}\frac{r_j}{r_i}(x_j - 1)\\
&\le w_\ell + b\alpha_2 \sum_{j=1}^{\ell-1}(x_j - 1).
\end{aligned} \tag{2.14}$$

Now, to construct the variable x_j for $j = 1, 2, \ldots, \ell - 1$ and $\ell = 2, \ldots, n$, we consider the real roots of the quadratic equations:

$$Z_\ell^2 - (w_\ell + \gamma)Z_\ell + \gamma w_\ell - b\alpha_2 \sum_{j=1}^{\ell-1}(w_j - w_\ell) = 0. \tag{2.15}$$

In particular, the trinomials in (2.15) have discriminant

$$\Delta_\ell \equiv (w_\ell + \gamma)^2 - 4\left(\gamma w_\ell - b\alpha_2 \sum_{j=1}^{\ell-1}(w_j - w_\ell)\right) = (w_\ell - \gamma)^2 + 4b\alpha_2 \sum_{j=1}^{\ell-1}(w_j - w_\ell).$$

Due to the hypotheses that $b > 1$, $\alpha_2 > 0$, and $\{w_i\}_{i=1}^n$ is a decreasing sequence of nonnegative numbers, the discriminant Δ_ℓ is nonnegative for all $\ell = 2, \ldots, n$, which yields that the quadratic equations in (2.15) have a root

$$Z_\ell = \frac{1}{2}(w_\ell + \gamma + \sqrt{\Delta_\ell}), \quad \ell = 2, \ldots, n. \tag{2.16}$$

Now, for $1 \le j \le \ell - 1$, we consider

$$x_j = 1 + \frac{w_j - w_\ell}{Z_\ell - \gamma} \Leftrightarrow x_j - 1 = \frac{w_j - w_\ell}{Z_\ell - \gamma}, \tag{2.17}$$

where Z_ℓ is given by (2.16). If $\sum_{j=1}^{\ell-1}(w_j - w_\ell) > 0$, it is clear by relation (2.16) that $Z_\ell > \frac{1}{2}(w_\ell + \gamma + |w_\ell - \gamma|) \ge \frac{1}{2}(w_\ell + \gamma - (w_\ell - \gamma)) = \gamma$, otherwise, $w_1 = \cdots = w_\ell > \gamma$ and (2.16) yields $Z_\ell = \frac{1}{2}(w_\ell + \gamma + |w_\ell - \gamma|) > \frac{1}{2}(w_\ell + \gamma - (w_\ell - \gamma)) = \gamma$. Both cases ensure $x_j - 1 \ge 0$ and x_j in (2.17) is well defined.

Moreover, by (2.15), we may write

$$b\alpha_2 \sum_{j=1}^{\ell-1}(w_j - w_\ell) = Z_\ell^2 - (w_\ell + \gamma)Z_\ell + \gamma w_\ell = (Z_\ell - \gamma)(Z_\ell - w_\ell). \tag{2.18}$$

Overall, we take $x_j - 1 \ge 0$, $j = 1, \ldots, \ell - 1$ from (2.17) and $b\alpha_2 \sum_{j=1}^{\ell-1}(w_j - w_\ell)$ from (2.18) and substitute them into the inequality (2.12). Hence, for $1 \le i \le \ell - 1$, we obtain

$$\begin{aligned} r_i(B) &\le \frac{1}{x_i}\left(w_i + b\alpha_2 \sum_{j=1}^{\ell-1}(x_j - 1) + \gamma(x_i - 1)\right) \\ &= \frac{1}{x_i}\left(w_i + b\alpha_2 \sum_{j=1}^{\ell-1}\frac{w_j - w_\ell}{Z_\ell - \gamma} + \gamma\frac{w_i - w_\ell}{Z_\ell - \gamma}\right) \\ &= \frac{1}{x_i}\left(w_i + \frac{(Z_\ell - \gamma)(Z_\ell - w_\ell)}{Z_\ell - \gamma} + \gamma\frac{w_i - w_\ell}{Z_\ell - \gamma}\right) \\ &= \frac{w_i(Z_\ell - \gamma) + (Z_\ell - \gamma)(Z_\ell - w_\ell) + \gamma(w_i - w_\ell)}{x_i(Z_\ell - \gamma)} \\ &= \frac{(Z_\ell - \gamma)Z_\ell + (w_i - w_\ell)(Z_\ell - \gamma + \gamma)}{x_i(Z_\ell - \gamma)} \\ &= \frac{(Z_\ell - \gamma)Z_\ell + (w_i - w_\ell)Z_\ell}{\frac{Z_\ell - \gamma + w_i - w_\ell}{Z_\ell - \gamma}(Z_\ell - \gamma)} = Z_\ell, \quad \ell = 2, \ldots, n. \end{aligned} \tag{2.19}$$

Similarly, for $\ell \le i \le n$ and $1 \le j \le \ell - 1$, we substitute the relations (2.18) and (2.17) into the inequality (2.14), which can be written as follows:

$$\begin{aligned} r_i(B) &\le w_\ell + b\alpha_2 \sum_{j=1}^{\ell-1}(x_j - 1) \\ &= w_\ell + \frac{b\alpha_2}{Z_\ell - \gamma}\sum_{j=1}^{\ell-1}(w_j - w_\ell) \\ &= w_\ell + \frac{(Z_\ell - \gamma)(Z_\ell - w_\ell)}{Z_\ell - \gamma} = Z_\ell. \end{aligned} \tag{2.20}$$

Thus, for $2 \le \ell \le n$ and $1 \le i \le n$, the inequalities (2.19) and (2.20) verify $r_i(B) \le Z_\ell$ and by Lemma 2.2, $\rho(A^3) = \rho(B) \le \max_{1\le i\le n}\{r_i(B)\} \le Z_\ell$. Thereby, the validity of (2.10) follows readily. □

Remark 1.

(i) For $1 \le i, j \le \ell - 1$, with $j \ne i$, inequality (2.12) is given by equality if and only if $x_i = x_j = 1$ or $\frac{r_j}{r_i} = b$ and $a_{ii}^{(3)} = \alpha_1$, $a_{ij}^{(3)} = \alpha_2$ when $x_i > 1$ and $x_j > 1$.

(ii) For $\ell \le i \le n$, $1 \le j \le \ell - 1$, inequality (2.14) is given by equality if and only if $w_i = w_\ell$ and $x_j = 1$ or $\frac{r_j}{r_i} = b$ and $a_{ij}^{(3)} = \alpha_2$ when $x_j > 1$.

Due to Proposition 2.1, $a_{ii}^{(3)} = \alpha_1$ and $a_{ij}^{(3)} = \alpha_2$ are satisfied whenever $a_{ii} = \mu$ and $a_{ij} = \nu$ for all $1 \le i, j \le n$, with $j \ne i$.

The inequality (2.10) proved in Theorem 2.4 is in fact strict. If we take into account Remark 1 and Lemma 2.3, then an equality for the spectral radius of a nonnegative and irreducible matrix is derived as shown in the following proposition.

Proposition 2.5. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \ge 0$ be irreducible and let the quantities $\mu, \nu, \gamma, Z_\ell, b$ with $b > 1$ and $r_i(A), w_i(A)$, $i = 1, \ldots, n$ satisfy the notations and assumptions of* Theorem 2.4. *Then, $\rho(A) = \sqrt[3]{Z_\ell}$ for some $\ell = 1, \ldots, n$ if and only if one of the following holds*:

(*i*) *For $\ell = 1$, $m_1(A) = \cdots = m_n(A)$, if A^3 is irreducible, otherwise, $PAP^T = \begin{pmatrix} 0_{n_1} & A_1 & 0 \\ 0 & 0_{n_2} & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$, for some permutation matrix P such that $m_{j_1}(A) = \cdots = m_{j_{n_1}}(A)$, $m_{j_{n_1+1}}(A) = \cdots = m_{j_{n_1+n_2}}(A)$, $m_{j_{n_1+n_2+1}}(A) = \cdots = m_{j_n}(A)$, where $\{j_1, \ldots, j_{n_1}\}$, $\{j_{n_1+1}, \ldots, j_{n_1+n_2}\}$, $\{j_{n_1+n_2+1}, \ldots, j_n\}$ form a partition of $\{1, \ldots, n\}$ corresponding to P.*

(*ii*) *For $\ell = 2, \ldots, n$, $w_1(A) = \cdots = w_n(A)$.*

Proof.

(i) For $\ell = 1$, the result is an immediate consequence of Lemma 2.3.

(ii) For $\ell = 2, \ldots, n$, we first suppose that $\rho(A) = \sqrt[3]{Z_\ell}$ with irreducible matrix $A \ge 0$ and we consider $B = U^{-1}A^3U \ge 0$, as constructed in the proof of Theorem 2.4. We then distinguish among two cases:

(a) If $A^3 \ge 0$ is irreducible, then B is also irreducible with $Z_\ell = \rho(A^3) = \rho(B) \le \max_{1\le i\le n}\{r_i(B)\} \le Z_\ell \Rightarrow \rho(B) = \max_{1\le i\le n}\{r_i(B)\} = Z_\ell$. By Lemma 2.2, $r_1(B) = \cdots = r_n(B) = Z_\ell$ and thus, inequalities (2.12) and (2.14) degenerate to equalities. If $w_1(A) > w_\ell(A)$, we consider the smallest integer $2 \le t \le \ell$ such that $w_t(A) = w_\ell(A)$. Clearly, $w_i(A) > w_\ell(A)$ for integers $1 \le i \le t - 1$, which implies that $x_i > 1$. Therefore, condition (i) of Remark 1 yields $a_{ii} = \mu$ and $a_{ij} = \nu$ for $1 \le i, j \le n$, $i \ne j$, which results in $r_1(A) = \cdots = r_n(A) = \mu + (n-1)\nu$. But then, we have the absurdity $b = 1$. Hence, $w_1(A) = w_\ell(A)$ and by case (ii) in Remark 1, we have $w_1(A) = \cdots = w_n(A)$.

(b) If A^3 is reducible, and so is B, there is a permutation matrix P such that

$$PA^3P^T = C_1 \oplus C_2 \oplus C_3$$

with irreducible matrices C_j, $j = 1, 2, 3$ and $\rho(A) = \sqrt[3]{\rho(C_j)}$ [4]. Clearly,

$$PBP^T = D^{-1}(C_1 \oplus C_2 \oplus C_3)D = B_1 \oplus B_2 \oplus B_3,$$

where $D = PUP^T$ is diagonal and $B_j \ge 0$, $j = 1, 2, 3$ are $n_j \times n_j$ irreducible matrices with $\rho(B_j) = \rho(C_j)$. By Lemma 2.2,

$$Z_\ell = \rho(B) = \rho(A)^3 = \rho(B_j) \le \max_{1\le i\le n_j}\{r_i(B_j)\} \le \max_{1\le i\le n}\{r_i(B)\} = Z_\ell \Rightarrow \rho(B_j) = \max_{1\le i\le n_j}\{r_i(B_j)\}$$

and since $B_j \in \mathcal{M}_{n_j}(\mathbb{R})$ is irreducible for any $j = 1, 2, 3$, we have $r_1(B_j) = \cdots = r_{n_j}(B_j) = Z_\ell$ for any $j = 1, 2, 3$. Due to permutational similarity, $r_1(B) = \cdots = r_n(B) = Z_\ell$ and (2.11) and (2.12) are equalities. Following the same arguments as previously, we conclude $w_1(A) = \cdots = w_n(A)$.

For the converse statement, if $w_1(A) = \cdots = w_n(A)$, we substitute into (2.8) and then, we obtain $Z_\ell = w_\ell(A) = w_1(A)$ for $1 \le \ell \le n$. Evidently, by Lemma 2.3, $\rho(A) = \max_{1\le i\le n}\sqrt[3]{w_i(A)} = \min_{1\le i\le n}\sqrt[3]{w_i(A)} = \sqrt[3]{w_\ell(A)} = \sqrt[3]{Z_\ell}$, $1 \le \ell \le n$. □

We compare the results stated in Theorem 2.4 to other established upper bounds at the next example.

Example 2.6. Consider the matrices $A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 5 & 0 & 0 & 5 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 4 & 0 & 4 & 0 & 0 & 0 \end{pmatrix}$ with spectral radii $\rho(A) = 4.2924$ and $\rho(B) = 6.8399$, respectively.

(i) Clearly, the (positive) row sums of matrix A are $r_1(A) = r_2(A) = 5$, $r_3(A) = 4$, $r_4(A) = 1$ and its average 4-row sums are $w_1(A) = 101$, $w_2(A) = 66.4$, $w_3(A) = 96.5$, $w_4(A) = 1$. To apply Theorem 2.4, we permute the second and third rows and columns of A taking the matrix

$$\hat{A} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\mu = 3$, $\nu = 2$, $r_1(\hat{A}) = 4$, $r_2(\hat{A}) = r_3(\hat{A}) = 5$, $r_4(\hat{A}) = 1$, $b = 5 > 1$, $\gamma = 183 - 5 \cdot 182 = -727$, and $w_1(\hat{A}) = 101$, $w_2(\hat{A}) = 96.5$, $w_3(\hat{A}) = 66.4$, $w_4(\hat{A}) = 1$. Obviously, the assumptions of Theorem 2.4 are ensured, and we can compute the quantities Z_ℓ given by (2.8), which are $Z_1 = 101$, $Z_2 = 101.443$, $Z_3 = 134.7246$, $Z_4 = 245.2064$. Then, $\rho(A) = \rho(\hat{A}) \le \sqrt[3]{101} = 4.657$.

(ii) We consider the irreducible matrix B whose power B^3 is reducible, and we take the permutation matrix P associated to the row-partition $\{1, 3\}, \{2, 5\}, \{4, 6\}$. Then, for $\ell = 1$, we obtain $P^TBP = \begin{pmatrix} 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$ such that $m_1(B) = m_3(B) = 4$, $m_2(B) = m_5(B) = 8$, and $m_4(B) = m_6(B) = 10$. Clearly, Proposition 2.5 holds, which implies that $\rho(B) = \sqrt[3]{Z_1} = \sqrt[3]{w_1(B)} = \sqrt[3]{320} = 6.8399$.

Table 1 displays a comparison among upper bounds for $\rho(A)$ and $\rho(B)$ computed by the different formulations presented in [2,7,11,15]. As one may observe, the upper bound for the spectral radius $\rho(A)$ computed by the expression of Theorem 2.4 appears to be a refinement, since it is sharper compared to the other values. Moreover, the exact value of $\rho(B)$ coincides with the quantity in Proposition 2.5.

Table 1: Comparison among different formulae of upper bounds for the spectral radius

Theorems for upper bounds	Value for *A*	Value for *B*
Duan and Zhou [5, Theorem 2.1]	5.0000	9.3899
Xing and Zhou [7, Theorem 2.1]	4.9671	10.0000
Lin and Zhou [6, Theorem 2.1]	4.7323	8.9443
Adam et al. [2, Theorem 4]	4.8173	17.5000
Theorem 2.4–Proposition 2.5	4.6570	6.8399
Spectral radius	4.2924	6.8399

3 Lower bound for the spectral radius of nonnegative matrices

In this section, we obtain a new result on the lower bound for the spectral radius of nonnegative matrices.

Theorem 3.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ with row sums $r_i(A) > 0$ for all $i = 1, \dots, n$, and average 4-row sums $w_1(A) \geq w_2(A) \geq \cdots \geq w_n(A) > 0$. Let ζ, η be the smallest diagonal and off-diagonal elements of A, respectively. Denote by $q = \min\left\{\frac{r_j(A)}{r_i(A)} : 1 \leq i, j \leq n\right\}$, and $\delta = \beta_1 - q\beta_2$, where β_1, β_2 are given by (2.2). Consider $w_n(A) > \delta$ and*

$$z_n = \frac{1}{2}(w_n(A) + \delta + \sqrt{\Delta_n}), \tag{3.1}$$

where

$$\Delta_n = (w_n(A) - \delta)^2 + 4q\beta_2 \sum_{j=1}^{n-1} (w_j(A) - w_n(A)). \tag{3.2}$$

Then,

$$\rho(A) \geq \sqrt[3]{z_n}. \tag{3.3}$$

Proof. To simplify the exposition of our calculations, we let $m_i = m_i(A)$, and $w_i = w_i(A)$ for $1 \leq i \leq n$.

If $\eta = 0$, by (2.2), $\delta = \beta_1 - q\beta_2 = \zeta^3$ arises, and then, the equality (3.1) leads to $z_n = w_n$ due to $w_n > \delta$; the result follows immediately from the monotonicity of the sequence of $\{w_i\}_{i=1}^n$ and Lemma 2.3. Consequently, in what follows, we assume $\eta > 0$.

Let $U = \operatorname{diag}(r_1x_1, r_2x_2, \dots, r_{n-1}x_{n-1}, r_n)$ be an $n \times n$ diagonal matrix, where $x_j \geq 1$ for $1 \leq j \leq n-1$ is a variable to be determined later and let $B = U^{-1}A^3U$. Due to similarity, A^3 and B have the same eigenvalues; hence, $\rho(A) = \sqrt[3]{\rho(A^3)} = \sqrt[3]{\rho(B)}$.

For $1 \leq i \leq n-1$, we derive

$$\begin{aligned}
r_i(B) &= \sum_{j=1}^{n-1} a_{ij}^{(3)} \frac{r_jx_j}{r_ix_i} + a_{in}^{(3)} \frac{r_n}{r_ix_i} = \frac{1}{x_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}^{(3)} \frac{r_jx_j}{r_i} + a_{ii}^{(3)}x_i + a_{in}^{(3)}\frac{r_n}{r_i}\right)\\
&= \frac{1}{x_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}^{(3)} \frac{r_jx_j}{r_i} + a_{ii}^{(3)}x_i + \sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} - \sum_{j=1}^{n-1} a_{ij}^{(3)}\frac{r_j}{r_i} + a_{in}^{(3)}\frac{r_n}{r_i}\right)\\
&= \frac{1}{x_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}^{(3)} \frac{r_jx_j}{r_i} + \sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} - \sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}^{(3)}\frac{r_j}{r_i} + a_{ii}^{(3)}x_i - a_{ii}^{(3)}\right)\\
&= \frac{1}{x_i}\left(\sum_{j=1}^{n} a_{ij}^{(3)}\frac{r_j}{r_i} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} a_{ij}^{(3)}\frac{r_j}{r_i}(x_j - 1) + a_{ii}^{(3)}(x_i - 1)\right).
\end{aligned}$$

Combining the equality in (1.4) and the left part of the inequalities in (2.3)–(2.4) with the assumption $\frac{r_j}{r_i} \geq q$ for $1 \leq j \leq n-1$ and $j \neq i$ and the definition $\delta = \beta_1 - q\beta_2$, the latter equality is formulated as follows:

$$r_i(B) \geq \frac{1}{x_i}\left(w_i + q\beta_2 \sum_{\substack{j=1\\ j\neq i}}^{n-1}(x_j - 1) + \beta_1(x_i - 1)\right) = \frac{1}{x_i}\left(w_i + q\beta_2\sum_{j=1}^{n-1}(x_j - 1) + \delta(x_i - 1)\right). \tag{3.4}$$

Furthermore, for $i = n$, we have

$$\begin{aligned} r_n(B) &= \sum_{j=1}^{n-1} a_{nj}^{(3)} \frac{r_j x_j}{r_n} + a_{nn}^{(3)} \sum_{j=1}^{n-1} a_{nj}^{(3)} \frac{r_j x_j}{r_n} + \sum_{j=1}^{n} a_{nj}^{(3)} \frac{r_j}{r_n} - \sum_{j=1}^{n-1} a_{nj}^{(3)} \frac{r_j}{r_n} \\ &= \sum_{j=1}^{n-1} a_{nj}^{(3)} \frac{r_j}{r_n}(x_j - 1) + \sum_{j=1}^{n} a_{nj}^{(3)} \frac{r_j}{r_n} = \sum_{j=1}^{n-1} a_{nj}^{(3)} \frac{r_j}{r_n}(x_j - 1) + w_n \\ &\geq q\beta_2 \sum_{j=1}^{n-1}(x_j - 1) + w_n. \end{aligned} \tag{3.5}$$

Now, it is easy to prove that the following quadratic equation:

$$z_n^2 - (w_n + \delta)z_n + \delta w_n - q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n) = 0 \tag{3.6}$$

has real roots, since its discriminant

$$\Delta_n \equiv (w_n + \delta)^2 - 4\left(\delta w_n - q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n)\right) = (w_n - \delta)^2 + 4q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n)$$

is a positive number, due to $\beta_2 \geq 0, c > 0$, and the monotonicity of the sequence $\{w_i\}_{i=1}^n$ of positive numbers. Hence, the quadratic equation in (3.6) has a positive real root

$$z_n = \frac{1}{2}(w_n + \delta + \sqrt{\Delta_n}), \tag{3.7}$$

which is used in the construction of

$$x_j = 1 + \frac{w_j - w_n}{z_n - \delta} \Leftrightarrow x_j - 1 = \frac{w_j - w_n}{z_n - \delta}, \tag{3.8}$$

for $1 \leq j \leq n-1$. If $\sum_{j=1}^{n-1}(w_j - w_n) > 0$, it is clear by relation (3.7) that $z_n > \frac{1}{2}(w_n + \delta + |w_n - \delta|) \geq \frac{1}{2}(w_n + \delta - (w_n - \delta)) = \delta$, otherwise, $w_1 = \cdots = w_n > \delta$ and (3.7) yields $z_n = \frac{1}{2}(w_n + \delta + |w_n - \delta|) > \frac{1}{2}(w_n + \delta - (w_n - \delta)) = \delta$. Both cases ensure $x_j - 1 \geq 0$ in (3.8).

Moreover, from (3.6), we derive

$$q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n) = z_n^2 - (w_n + \delta)z_n + w_n\delta = (z_n - \delta)(z_n - w_n). \tag{3.9}$$

For $1 \leq i, j \leq n-1$, substitute $q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n)$ from (3.9) and $x_j - 1 \geq 0$ from (3.8) into the inequality (3.4), which can be written as follows:

$$\begin{aligned} r_i(B) &\geq \frac{1}{x_i}\left(w_i + q\beta_2 \sum_{j=1}^{n-1}(x_j - 1) + \delta(x_i - 1)\right) \\ &= \frac{1}{x_i}\left(w_i + q\beta_2 \sum_{j=1}^{n-1} \frac{w_j - w_n}{z_n - \delta} + \delta \frac{w_i - w_n}{z_n - \delta}\right) \\ &= \frac{1}{x_i}\left(w_i + \frac{(z_n - \delta)(z_n - w_n)}{z_n - \delta} + \delta \frac{(w_i - w_n)}{z_n - \delta}\right) \\ &= \frac{w_i(z_n - \delta) + (z_n - \delta)(z_n - w_n) + \delta(w_i - w_n)}{x_i(z_n - \delta)} \\ &= \frac{(z_n - \delta)(z_n + w_i - w_n) + \delta(w_i - w_n)}{x_i(z_n - \delta)} \\ &= \frac{z_n(z_n - \delta) + z_n(w_i - w_n)}{x_i(z_n - \delta)} \end{aligned} \tag{3.10}$$

$$= \frac{z_n(z_n - \delta + w_i - w_n)}{x_i(z_n - \delta)}$$
$$= \frac{z_n(z_n - \delta + w_i - w_n)}{\frac{z_n - \delta + w_i - w_n}{z_n - \delta}(z_n - \delta)} = z_n.$$

Moreover, substitute $x_j - 1$ from (3.8) and $q\beta_2 \sum_{j=1}^{n-1}(w_j - w_n)$ from (3.9) into the inequality (3.5), which gives

$$r_n(B) \geq w_n + q\beta_2 \sum_{j=1}^{n-1} \frac{w_j - w_n}{z_n - \delta} = w_n + \frac{(z_n - \delta)(z_n - w_n)}{z_n - \delta} = z_n. \tag{3.11}$$

Hence, for all $1 \leq i \leq n$, the inequalities (3.10) and (3.11) confirm $r_i(B) \geq z_n$. By Lemma 2.2, $\rho(A^3) = \rho(B) \geq \min_{1 \leq i \leq n}\{r_i(B)\} \geq z_n$, verifying the validity of (3.3). □

Remark 2.
(i) For $1 \leq i, j \leq n - 1$, with $j \neq i$ inequality (3.4) is given by equality if and only if $x_i = x_j = 1$ or $\frac{r_j}{r_i} = q$ and $a_{ii}^{(3)} = \beta_1$, $a_{ij}^{(3)} = \beta_2$ when $x_i > 1$ and $x_j > 1$.
(ii) For $1 \leq i \leq n$ and $1 \leq j \leq n - 1$ inequality (3.5) is given by equality if and only if $x_j = 1$ or $\frac{r_j}{r_i} = q$ and $a_{ii}^{(3)} = \beta_1$, $a_{nj}^{(3)} = \beta_2$ when $x_j > 1$.

Due to Proposition 2.1, $a_{ii}^{(3)} = \beta_1$ and $a_{ij}^{(3)} = \beta_2$ are satisfied whenever $a_{ii} = \zeta$ and $a_{ij} = \eta$ for all $1 \leq i, j \leq n$, with $j \neq i$.

Using Remark 2, Theorem 3.1 can be reduced to the next proposition, which yields the equality cases of the lower bound for the spectral radius of nonnegative and irreducible matrices.

Proposition 3.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ be irreducible and let the quantities $\zeta, \eta, \delta, z_n, q$, with $q > 1$ and $r_i(A), w_i(A), i = 1, \ldots, n$ satisfy the notations and assumptions of* Theorem 3.1. *Then, $\rho(A) = \sqrt[3]{z_n}$ if and only if one of the following holds*:

(*i*) *If $\eta = 0$, then $m_1(A) = \cdots = m_n(A)$, if A^3 is irreducible, otherwise, $PAP^T = \begin{pmatrix} 0_{n_1} & A_1 & 0 \\ 0 & 0_{n_2} & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$, for some permutation matrix P such that $m_{j_1}(A) = \cdots = m_{j_{n_1}}(A)$, $m_{j_{n_1+1}}(A) = \cdots = m_{j_{n_1+n_2}}(A)$, $m_{j_{n_1+n_2+1}}(A) = \cdots = m_{j_n}(A)$, where $\{j_1, \ldots, j_{n_1}\}$, $\{j_{n_1+1}, \ldots, j_{n_1+n_2}\}$, $\{j_{n_1+n_2+1}, \ldots, j_n\}$ form a partition of $\{1, \ldots, n\}$ corresponding to P.*
(*ii*) *If $\eta > 0$, $w_1(A) = \cdots = w_n(A)$.*

At the next example, we test the lower bound for the spectral radius proved in Theorem 3.1 compared to the ones stated in [2,5–7].

Example 3.3. Consider matrices $A = \begin{pmatrix} 6 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}$ and B as in Example 2.6 with spectral radii $\rho(A) = 8$ and $\rho(B) = 6.8399$, respectively.

(i) The (positive) row sums of matrix A are $r_1(A) = 12$, $r_2(A) = 5$, $r_3(A) = r_4(A) = 6$ and its average 4-row sums are $w_1(A) = 590.3333$, $w_2(A) = 293.6$, $w_3(A) = w_4(A) = 476.6667$. If we permute the second and fourth rows and columns of A, then we take

$$\widetilde{A} = \begin{pmatrix} 6 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

with $\zeta = \eta = \delta = 0, q = 0.4167$, and $w_1(\widetilde{A}) = 590.3333$, $w_2(\widetilde{A}) = w_3(\widetilde{A}) = 476.6667$, $w_4(\widetilde{A}) = 293.6$. Obviously, the assumptions of Theorem 3.1 are ensured and $z_n = w_4(\widetilde{A}) = 293.6$ given by (3.1). Then, $\rho(A) = \rho(\widetilde{A}) \geq \sqrt[3]{293.6} = 6.6464$.

(ii) The irreducible matrix B with $\eta = 0$ and B^3 reducible infers that $m_1(B) = m_3(B) = 4$, $m_2(B) = m_5(B) = 8$, and $m_4(B) = m_6(B) = 10$. Apparently, Proposition 3.2 holds and the spectral radius of B is identified with $\rho(B) = \sqrt[3]{z_6} = \sqrt[3]{w_6} = \sqrt[3]{320} = 6.8399$.

Table 2 records the lower bounds for $\rho(A)$ and $\rho(B)$ computed by the formulations presented in [2,5–7]. As observed, the lower bound for $\rho(A)$ given by Theorem 3.1 is sharper than the other values and $\rho(B)$ equals the quantity in Proposition 3.2.

4 Application on generalized Fibonacci matrices

This section is devoted to applications of Theorems 2.4 and 3.1 and also of the formulations in [5–7] to special matrices associated with the Fibonacci sequence. This famous sequence has grown over the years into a vital tool for applied mathematics and computer science, with various applications to certain sorting algorithms and maximal network flow problems. For instance, in graph theory, the problem of enumerating all perfect matchings in a bipartite graph is closely related to generalized Fibonacci numbers [10,11].

We consider a generalization of the Fibonacci sequence called the *generalized k-step Fibonacci sequence* abbreviated by $(f_n^{\{k\}}(c_1, c_2, \ldots, c_k))_{n\in\mathbb{N}}$ with respect to $k \geq 2$ real numbers $c_1 > 0, c_2, \ldots, c_k \geq 0$. For every $n > k$, the *nth generalized k-Fibonacci number* is defined recursively as a linear combination of the preceding k terms

$$f_n^{\{k\}} = \sum_{j=1}^{k} c_j f_{n-j}^{\{k\}} = c_1 f_{n-1}^{\{k\}} + c_2 f_{n-2}^{\{k\}} + \cdots + c_k f_{n-k}^{\{k\}}, \tag{4.1}$$

with initial values $f_1^{\{k\}} = \cdots = f_k^{\{k\}} = 1$, [12,13]. We notice that for $k = 2$, the sequence $(f_n^{\{2\}}(1,1))_{n\in\mathbb{N}}$ reduces to the classical Fibonacci sequence, and for $k = 3$, we obtain the Tribonacci sequence $(f_n^{\{3\}}(1,1,1))_{n\in\mathbb{N}}$, followed by the Tetranacci sequence with $k = 4$, and so on. Now, let the vector $\mathbf{F}_n^{\{k\}} = \begin{pmatrix} f_n^{\{k\}} & f_{n-1}^{\{k\}} & \cdots & f_{n-k+1}^{\{k\}} \end{pmatrix}^T \in \mathcal{M}_{k,1}(\mathbb{R})$, $k \geq 2$ and $n > k$ with $\mathbf{F}_k^{\{k\}} = (1 \;\; 1 \;\; \cdots \;\; 1)^T$ to gather the initial values of the sequence, then the generalized k-Fibonacci sequence in (4.1) can be also defined by the recurrence relation:

$$\mathbf{F}_n^{\{k\}} = Q_k(c_1, \ldots, c_k)\mathbf{F}_{n-1}^{\{k\}}, \tag{4.2}$$

associated with the *generalized k-Fibonacci matrix*

$$Q_k(c_1, \ldots, c_k) = \begin{pmatrix} \mathbf{c} & c_k \\ I_{k-1} & 0_{k-1,1} \end{pmatrix} \in \mathcal{M}_k(\mathbb{R}), \quad \mathbf{c} = (c_1 \;\; \cdots \;\; c_{k-1}). \tag{4.3}$$

Observe that by an inductive argument, (4.2) may lead to $\mathbf{F}_n^{\{k\}} = Q_k^{n-k}(c_1, \ldots, c_k)\mathbf{F}_k^{\{k\}}$.

The matrix $Q_k(c_1, \ldots, c_k)$ has been widely used in the design of many encryption/decryption algorithms, which aim to provide secure and robust schemes of minimum vulnerability to attack. Due to the interesting

Table 2: Comparison among several formulae of lower bounds for the spectral radius

Theorems for lower bounds	Values for A	Values for B
Duan and Zhou [5, Theorem 2.3]	5.0000	4.0000
Xing and Zhou [7, Theorem 2.3]	5.6000	4.0000
Lin and Zhou [6, Theorem 2.3]	6.2290	5.6569
Adam et al. [2, Theorem 10]	5.7652	3.2000
Theorem 3.1–Proposition 3.2	6.6464	6.8399
Spectral radius	8.0000	6.8399

properties, it enjoys (see [12–15]), and it leads to faster implementations reducing the time as well as the space complexity of the security process. Most notably, it is an irreducible and primitive matrix with a simple (without multiplicity) and unique eigenvalue of maximum magnitude, its spectral radius $\rho(Q_k(c_1,\dots,c_k)) > 0$.

In [11], the authors located $\rho(Q_k(1,\dots,1))$ outside the unit disk, especially lying between $(2-k^{-1})^{1/2}$ and 2. The current section aims to investigate and present different bounding regions for the spectral radius $\rho(Q_k(c_1,\dots,c_k))$, by applying formulas that use and extend the row and average row sums of nonnegative matrices. In the remainder of this work, we assume $c_i \geq 1$ for all $i = 1,\dots,k$ and we denote $C = \sum_{i=1}^{k} c_i$ and $Q_k \coloneqq Q_k(c_1,\dots,c_k)$ for reasons of simplicity. The assumption $c_i \geq 1$, $i = 1,\dots,k$ verifies the ordering of the row sums of Q_k

$$C = r_1(Q_k) > r_2(Q_k) = r_3(Q_k) = \cdots = r_k(Q_k) = 1, \tag{4.4}$$

which combined with Lemma 2.2 clearly derives that $1 \leq \rho(Q_k) \leq C$. Due to the relation (4.4), we can immediately apply Duan and Zhou's formulae in [5] to bound $\rho(Q_k)$, which lead to the following result.

Proposition 4.1. *Let $k \geq 2$ nonnegative real numbers $c_1, c_2, \dots, c_k \geq 1$ and denote by $\nu = \max_{2\leq i\leq k}\{c_i\}$.*
(*i*) *If $k \geq 3$, then*

$$1 \leq \rho(Q_k) \leq \frac{1+c_1-\nu}{2} + \sqrt{\left(\frac{1-c_1+\nu}{2}\right)^2 + \nu(C-1)}\,. \tag{4.5}$$

(*ii*) *If $k = 2$, then*

$$\sqrt{c_1+c_2} \leq \rho(Q_2) \leq \frac{1+c_1-c_2}{2} + \sqrt{\left(\frac{c_1+c_2-1}{2}\right)^2 + c_2^2}\,.$$

Proof. The lower bound for $\rho(Q_k)$ is computed by applying Theorem 2.2 in [5]

$$\psi_k = \frac{r_k+\zeta-\eta}{2} + \sqrt{\left(\frac{r_k-\zeta+\eta}{2}\right)^2 - \eta k r_k + \eta\sum_{j=1}^{k} r_j} = \begin{cases}\sqrt{c_1+c_2}, & k=2\\ 1, & k\geq 3,\end{cases}$$

where $r_k \coloneqq r_k(Q_k) > 0$ is the kth row sum of Q_k, $k \geq 2$ satisfying (4.4). Note that the minimum diagonal and off diagonal elements of Q_k for $k = 2$ are $\zeta = 0$, $\eta = 1$, respectively, whereas for $k \geq 3$ are $\zeta = \eta = 0$.

Now, the upper bound for $\rho(Q_k)$ is derived applying Theorem 2.1 in [5]. Therefore, $\rho(Q_k) \leq \min\{\Psi_\ell : 1 \leq \ell \leq k\}$, where

$$\begin{aligned}\Psi_\ell &= \frac{r_\ell+\mu-\nu}{2} + \sqrt{\left(\frac{r_\ell-\mu+\nu}{2}\right)^2 - \nu\ell r_\ell + \nu\sum_{j=1}^{\ell} r_j}\\ &= \begin{cases} C, & \ell = 1\\ \dfrac{1+c_1-\nu}{2} + \sqrt{\left(\dfrac{1-c_1+\nu}{2}\right)^2 + \nu(C-1)}\,, & \ell \neq 1,\end{cases}\end{aligned}$$

since $\mu = c_1 \geq 1$ and $\nu = \max_{2\leq i\leq k}\{c_i\} \geq 1$ are the largest diagonal and off-diagonal elements of Q_k, respectively. Finally, we obtain the desired upper bound for $\rho(Q_k)$, due to the inequality

$$\frac{1+c_1-\nu}{2} + \frac{1}{2}\sqrt{(1-c_1+\nu)^2 + 4\nu(C-1)} < C. \qquad \square$$

We notice that the vector $\mathbf{m}(Q_k)$ with components $m_i(Q_k)$, $i = 1,\dots,k$ as in (1.1) can be written as follows:

$$\mathbf{m}(Q_k) = D_k^{-1} Q_k^2 E_k \in \mathcal{M}_{k,1}(\mathbb{R}), \tag{4.6}$$

where the diagonal matrix $D_k = \mathrm{diag}(C, 1, \ldots, 1) \in \mathcal{M}_k(\mathbb{R})$ and $E_k \in \mathcal{M}_{k,1}(\mathbb{R})$ is the vector with all entries equal to 1. Consequently,

$$\begin{aligned} m_1(Q_k) &= \frac{1}{C}\left(c_1 \sum_{j=1}^{k} c_j + \sum_{j=2}^{k} c_j\right) = 1 + c_1\left(1 - \frac{1}{C}\right), \\ m_2(Q_k) &= C, \\ m_3(Q_k) &= m_4(Q_k) = \cdots = m_k(Q_k) = 1. \end{aligned} \tag{4.7}$$

We utilize Xing and Zhou's formulae in [7] to bound $\rho(Q_k)$, thus yielding the next statement. In what follows, we consider $k \geq 3$, since the spectral radius of Q_k for $k = 2$ can be easily computed.

Proposition 4.2. *Let $k \geq 3$ nonnegative real numbers $c_1, c_2, \ldots, c_k \geq 1$ and $v = \max_{2 \leq i \leq k}\{c_i\}$. Then,*

$$1 \leq \rho(Q_k) \leq \min\{\widetilde{\Psi}_1, \widetilde{\Psi}_2, \widetilde{\Psi}_3\}, \tag{4.8}$$

where

$$\begin{aligned} \widetilde{\Psi}_1 &= C, \\ \widetilde{\Psi}_2 &= \frac{1 + c_1(2 - \frac{1}{C}) - vC}{2} + \frac{1}{2}\sqrt{\left(1 - \frac{c_1}{C} + vC\right)^2 + 4v(C - c_1)(C - 1)}, \\ \widetilde{\Psi}_3 &= \frac{1 + c_1 - vC}{2} + \frac{1}{2}\sqrt{(1 - c_1 + vC)^2 + 4v(C + c_1)(C - 1)}. \end{aligned}$$

Proof. Evidently, $1 < m_1(Q_k) < C$ arises from (4.7), which verifies

$$m_2(Q_k) > m_1(Q_k) > m_3(Q_k) = m_4(Q_k) = \cdots = m_k(Q_k). \tag{4.9}$$

Before applying the formulae treated in [7], we need to permute Q_k for its average 2-row sums to be decreasingly ordered. Consider the permutation matrix $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \vdots \\ 0 & \cdots & I_{k-2} \end{pmatrix} \in \mathcal{M}_k(\mathbb{R})$ and $\widetilde{Q}_k = P_1 Q_k P_1^T$. Then, the average 2-row sums of $\widetilde{Q}_k$ are the entries of the vector $\mathbf{m}(\widetilde{Q}_k) = P_1\mathbf{m}(Q_k) = (m_2(Q_k) \;\; m_1(Q_k) \;\; m_3(Q_k) \;\; \cdots \;\; m_k(Q_k))^T$, which by (4.9), verify $m_1(\widetilde{Q}_k) > m_2(\widetilde{Q}_k) > m_3(\widetilde{Q}_k) = \cdots = m_k(\widetilde{Q}_k) = 1$.

The lower bound for $\rho(Q_k) = \rho(\widetilde{Q}_k)$ given by [7, Theorem 2.3] is expressed as follows:

$$\widetilde{\psi}_k = \frac{m_k(\widetilde{Q}_k) + \zeta - q\eta}{2} + \sqrt{\left(\frac{m_k(\widetilde{Q}_k) - \zeta + q\eta}{2}\right)^2 + q\eta\sum_{j=1}^{k-1}(m_j(\widetilde{Q}_k) - m_k(\widetilde{Q}_k))} = 1,$$

since the minimum diagonal and off diagonal elements of $\widetilde{Q}_k$ for $k \geq 3$ are $\zeta = \eta = 0$. Also, $q = \min_{1 \leq i,j \leq k}\{r_j(\widetilde{Q}_k)/r_i(\widetilde{Q}_k)\} = \min\left\{C, \frac{1}{C}, 1\right\} = 1/C$.

The desired upper bounds are derived applying [7, Theorem 2.1]. Therefore, $\rho(Q_k) \leq \min\{\widetilde{\Psi}_\ell : 1 \leq \ell \leq k\}$, where

$$\widetilde{\Psi}_\ell = \frac{m_\ell(\widetilde{Q}_k) + \mu - bv}{2} + \sqrt{\left(\frac{m_\ell(\widetilde{Q}_k) - \mu + bv}{2}\right)^2 + bv\sum_{j=1}^{\ell-1}(m_j(\widetilde{Q}_k) - m_\ell(\widetilde{Q}_k))}.$$

Note that $\mu = c_1$ and $v = \max_{2 \leq i \leq k}\{c_i\}$ are the largest diagonal and off-diagonal elements of $\widetilde{Q}_k$, respectively, and $b = \max_{1 \leq i,j \leq k}\{r_j(\widetilde{Q}_k)/r_i(\widetilde{Q}_k)\} = \max\left\{C, \frac{1}{C}, 1\right\} = C$. ◻

Analogously to (4.6) and according to the definition (1.2), the vector

$$\mathbf{s}(Q_k) = D_k^{-1}Q_k^3E_k = D_k^{-1}Q_kD_k\mathbf{m}(Q_k) = \begin{pmatrix} c_1 & \frac{c_2}{C} & \cdots & \frac{c_{k-1}}{C} & \frac{c_k}{C} \\ C & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}\mathbf{m}(Q_k) \tag{4.10}$$

has entries the average 3-row sums of Q_k, that is,

$$\begin{aligned} s_1(Q_k) &= \frac{1}{C}\left((c_1^2 + c_2)\sum_{j=1}^{k} c_j + c_1\sum_{j=2}^{k} c_j + \sum_{j=3}^{k} c_j\right) = (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1 = s_1, \\ s_2(Q_k) &= c_1\sum_{j=1}^{k} c_j + \sum_{j=2}^{k} c_j = c_1C + C - c_1, \\ s_3(Q_k) &= C, \\ s_4(Q_k) &= s_5(Q_k) = \cdots = s_k(Q_k) = 1. \end{aligned} \tag{4.11}$$

In the following proposition, we apply Lin and Zhou's statements in [6] to derive lower and upper bounds for $\rho(Q_k)$.

Proposition 4.3. *Let $k \geq 3$ nonnegative real numbers $c_1, c_2, \ldots, c_k \geq 1$, $\nu = \max_{2\leq i\leq k}\{c_i\}$ and assume $C > \frac{c_1^2 + c_1 + (k-2)\nu^2}{1 + (2\nu+1)c_1 + (k-2)\nu^2}$.*

(*i*) *If $k \geq 4$, then we have*

$$1 \leq \rho(Q_k) \leq \min_{1\leq \ell\leq k}\{\sqrt{\hat{\Psi}_\ell}\}, \tag{4.12}$$

where

$$\begin{aligned} \hat{\Psi}_1 &= C + c_1(C-1), \\ \hat{\Psi}_2 &= \begin{cases} \frac{1}{2}\left(C + \lambda_1 + \sqrt{(C-\lambda_1)^2 + 4\lambda_2c_1C(C-1)}\right), & \text{if } c_1^2 + c_1 + c_2 < C \\ \frac{1}{2}\left(s_1 + \lambda_1 + \sqrt{(s_1-\lambda_1)^2 + 4\lambda_2C[c_1(C-1) + C - s_1]}\right), & \text{otherwise} \end{cases} \\ \hat{\Psi}_3 &= \begin{cases} \frac{1}{2}\left(s_1 + \lambda_1 + \sqrt{(s_1-\lambda_1)^2 + 4\lambda_2C[c_1(C-1) + 2(C-s_1)]}\right), & \text{if } c_1^2 + c_1 + c_2 < C \\ \frac{1}{2}\left(C + \lambda_1 + \sqrt{(C-\lambda_1)^2 + 4\lambda_2C[c_1(C-1) - (C-s_1)]}\right), & \text{otherwise} \end{cases} \\ \hat{\Psi}_4 &= \cdots = \hat{\Psi}_k = \frac{1}{2}\left(1 + \lambda_1 + \sqrt{(1-\lambda_1)^2 + 4\lambda_2C[(C-1)(c_1+2) + (s_1-1)]}\right), \end{aligned}$$

with s_1 given by (4.11) *and $\lambda_1 = (c_1-\nu)^2 + \lambda_2(1-C)$, $\lambda_2 = 2c_1\nu + (k-2)\nu^2$.*

(ii) *If $k = 3$, then*

$$\sqrt{\hat{\psi}_3} \leq \rho(Q_3) \leq \min\{\sqrt{\hat{\Psi}_1}, \sqrt{\hat{\Psi}_2}, \sqrt{\hat{\Psi}_3}\}, \tag{4.13}$$

where $\hat{\psi}_3 = \begin{cases} c_1^2 + c_1 + c_2 + \frac{c_3 - c_1^2}{c_1 + c_2 + c_3}, & \textit{if } c_1^2 < c_3 \\ c_1 + c_2 + c_3, & \textit{otherwise}. \end{cases}$

Proof. Initially, we study the ordering of the quantities $s_i(Q_k)$ in (4.11) for $i = 1, \ldots, k \geq 3$ by distinguishing among two cases:

(i) Assume $c_1^2 + c_1 + c_2 < C$, then $1 < (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1 < C$ implies

$$s_2(Q_k) > s_3(Q_k) > s_1(Q_k) > s_4(Q_k) = \cdots = s_k(Q_k) = 1. \tag{4.14}$$

Consider the permutation matrix $P_2 = \begin{pmatrix} P & 0_{3,k-3} \\ 0_{k-3,3} & I_{k-3} \end{pmatrix}$ with $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, and $\widehat{Q}_k = P_2 Q_k P_2^T$. Then $\mathbf{s}(\widehat{Q}_k) = P_2\mathbf{s}(Q_k) = (s_2(Q_k) \;\; s_3(Q_k) \;\; s_1(Q_k) \;\; s_4(Q_k) \;\; \cdots \;\; s_k(Q_k))^T$ has entries the average 3-row sums, which satisfy (4.14). Thus, we verify

$$s_1(\widehat{Q}_k) > s_2(\widehat{Q}_k) > s_3(\widehat{Q}_k) > s_4(\widehat{Q}_k) = \cdots = s_k(\widehat{Q}_k) = 1.$$

(ii) On the other hand, if $c_1^2 + c_1 + c_2 \geq C$, then $s_1 = (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1 \geq C > 1$. Moreover,

$$\begin{aligned} Cc_1\left(1 - \frac{1}{C}\right) + C > (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1 &\Leftrightarrow \\ Cc_1\left(1 - \frac{1}{C}\right) + C - 1 > (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) &\Leftrightarrow \\ Cc_1\left(1 - \frac{1}{C}\right) + C\left(1 - \frac{1}{C}\right) > (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) &\Leftrightarrow \\ (Cc_1 + C - c_1^2 - c_1 - c_2)\left(1 - \frac{1}{C}\right) > 0 \Leftrightarrow (C-1)c_1 + C > c_1^2 + c_2. \end{aligned}$$

The validity of the last inequality leads to the inequalities

$$s_2(Q_k) > s_1(Q_k) \geq s_3(Q_k) > s_4(Q_k) = \cdots = s_k(Q_k) = 1. \tag{4.15}$$

Take P_1 as in the proof of Proposition 4.2, then $\mathbf{s}(\widetilde{Q}_k) = \mathbf{s}(P_1 Q_k P_1^T) = P_1\mathbf{s}(Q_k) = (s_2(Q_k) \;\; s_1(Q_k) \;\; s_3(Q_k) \;\; s_4(Q_k) \;\; \cdots \;\; s_k(Q_k))^T$
is such that (4.15) holds. Apparently,

$$s_1(\widetilde{Q}_k) > s_2(\widetilde{Q}_k) \geq s_3(\widetilde{Q}_k) > s_4(\widetilde{Q}_k) = \cdots = s_k(\widetilde{Q}_k) = 1.$$

For $k \geq 3$, either cases (i) or (ii) may hold. In both occurrences, the permuted matrices $\widehat{Q}_k$ or $\widetilde{Q}_k$ have the same maximum and minimum diagonal and off-diagonal elements, namely, $\mu = c_1$, $\nu = \max_{2\leq i\leq k}\{c_i\}$, $\zeta = \eta = 0$ and also $b = C > 1$, $q = \frac{1}{C}$. The hypothesis $C > \frac{c_1 + c_1^2 + (k-1)\nu^2}{1 + (2\nu+1)c_1 + (k-2)\nu^2}$ ensures the assumptions

$$s_1(\widehat{Q}_k) = s_1(\widetilde{Q}_k) = c_1C + C - c_1 > \lambda_1 \quad \text{and} \quad s_k(\widehat{Q}_k) = s_k(\widetilde{Q}_k) = 1 > 0 = \theta_1$$

made in Theorems 2.1 and 2.3 in [6]. Then,

$$\hat{\psi}_k = \frac{s_k + \theta_1}{2} + \sqrt{\left(\frac{s_k - \theta_1}{2}\right)^2 + (2\zeta\eta q + (k-2)\eta^2 q)\sum_{j=1}^{k-1}(s_j - s_k)} \tag{4.16}$$

yields the lower bounds

$$\hat{\psi}_3 = \begin{cases} (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1, & c_1^2 < c_3 \\ C, & c_1^2 \geq c_3 \end{cases}, \quad \text{and} \quad \hat{\psi}_k = 1, \quad k \geq 4.$$

Likewise, the relation

$$\hat{\Psi}_\ell = \frac{s_\ell + \lambda_1}{2} + \sqrt{\left(\frac{s_\ell - \lambda_1}{2}\right)^2 + (2\mu\nu b + (k-2)\nu^2 b)\sum_{j=1}^{\ell-1}(s_j - s_\ell)}, \tag{4.17}$$

where $s_\ell := s_\ell(\widehat{Q}_k)$ or $s_\ell := s_\ell(\widetilde{Q}_k)$ depending on whether case (i) or (ii) is used, derives the desired upper bounds $\hat{\Psi}_\ell$, $1 \leq \ell \leq k$. Therefore,

$$\sqrt{\hat{\psi}_k} \leq \rho(Q_k) = \rho(\widehat{Q}_k) = \rho(\widetilde{Q}_k) \leq \min\{\sqrt{\hat{\Psi}_\ell} : 1 \leq \ell \leq k\}.$$ □

Finally, following similar notation as previously discussed, we can express our new quantity (1.3) by the vector

$$\mathbf{w}(Q_k) = D_k^{-1}Q_k^4E_k = D_k^{-1}Q_kD_k\mathbf{s}(Q_k) \in \mathcal{M}_{k,1}(\mathbb{R}), \tag{4.18}$$

with components the average 4-row sums of Q_k

$$\begin{aligned}
w_1(Q_k) &= \frac{1}{C}\left[(c_1^3 + 2c_1c_2 + c_3)\sum_{j=1}^{k} c_j + (c_1^2 + c_2)\sum_{j=2}^{k} c_j + c_1\sum_{j=3}^{k} c_j + \sum_{j=4}^{k} c_j\right] \\
&= \begin{cases} (c_1^3 + c_1^2 + 2c_1c_2 + c_1 + c_2 + c_3)\left(1 - \dfrac{1}{C}\right) + 1, & k \geq 3 \\ c_1^3 + 2c_1c_2 + (c_1^2 + c_2)\dfrac{c_2}{C}, & k = 2 \end{cases} \\
w_2(Q_k) &= C + (c_1^2 + c_1 + c_2)(C - 1), \\
w_3(Q_k) &= C + c_1(C - 1), \\
w_4(Q_k) &= C, \\
w_5(Q_k) &= w_6(Q_k) = \cdots = w_k(Q_k) = 1.
\end{aligned} \tag{4.19}$$

Proposition 4.4. *Let $k \geq 3$ real numbers $c_1, c_2, \ldots, c_k \geq 1$ and $v = \max_{2\leq i\leq k}\{c_i\}$. Assume $\frac{(v-c_1)^3+1}{c_1^2+c_1+c_2+\alpha} + 1 > \frac{1}{C}$, where $\alpha = (k^2 - 3k + 2)v^3 + 3(k-1)c_1v^2 + c_1^3$. Then,*

$$\sqrt[3]{\widetilde{w}_k} \leq \rho(Q_k) \leq \min_{1\leq \ell\leq k}\{\sqrt[3]{Z_\ell}\}, \tag{4.20}$$

where

$$\begin{aligned}
Z_1 &= \widetilde{w}_1 = C + (c_1^2 + c_1 + c_2)(C - 1), \\
Z_\ell &= \frac{\widetilde{w}_\ell + \gamma}{2} + \sqrt{\left(\frac{\widetilde{w}_\ell - \gamma}{2}\right)^2 + \frac{C[\gamma + (v - c_1)^3]}{1 - C}\left(\sum_{i=1}^{\ell-1}\widetilde{w}_i - (\ell - 1)\widetilde{w}_\ell\right)}, \quad 2 \leq \ell \leq 5,
\end{aligned}$$

and $Z_\ell = Z_5$ for $5 < \ell \leq k$ *with*

$$\begin{aligned}
\widetilde{w}_2 &= \max\left\{(c_1^3 + c_1^2 + 2c_1c_2 + c_1 + c_2 + c_3)\left(1 - \frac{1}{C}\right) + 1,\ C + c_1(C - 1)\right\}, \\
\widetilde{w}_3 &= \min\left\{(c_1^3 + c_1^2 + 2c_1c_2 + c_1 + c_2 + c_3)\left(1 - \frac{1}{C}\right) + 1,\ C + c_1(C - 1)\right\}, \\
\widetilde{w}_4 &= C, \quad \widetilde{w}_5 = 1 \quad \textit{and} \quad \gamma = \alpha(1 - C) - C(v - c_1)^3.
\end{aligned}$$

Proof. First, we investigate the ordering of the quantities $w_i(Q_k)$ in (4.19) for $i = 1, \ldots, k \geq 3$. It is immediate to have $w_2(Q_k) > w_3(Q_k) > w_4(Q_k) > w_5(Q_k) = \cdots = w_k(Q_k) = 1$, so we will show $w_2(Q_k) > w_1(Q_k)$. Indeed

$$\begin{aligned}
w_2(Q_k) &= \sum_{j=1}^{k} c_j(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + C \\
&= c_1(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + \sum_{j=2}^{k} c_j(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + C \\
&> c_1(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + (c_2 + c_3)(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1 \\
&= c_1(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + [c_1^2c_2 + c_1c_2 + c_2^2 + c_3(c_1^2 + c_1 + c_2)]\left(1 - \frac{1}{C}\right) + 1 \\
&> c_1(c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + (c_1c_2 + c_1 + c_2 + c_3)\left(1 - \frac{1}{C}\right) + 1 = w_1(Q_k),
\end{aligned}$$

since $c_j \geq 1$ for all $j = 1, \dots, k$. Therefore, for $k \geq 3$, we deduce

$$w_2(Q_k) > \max\{w_1(Q_k), w_3(Q_k)\} > \min\{w_1(Q_k), w_3(Q_k)\} > w_4(Q_k) > w_5(Q_k) = \cdots = w_k(Q_k).$$

Next we distinguish among two cases for $w_1(Q_k)$ and $w_3(Q_k)$:

(i) If $w_1(Q_k) \geq w_3(Q_k)$, we employ $\widetilde{Q}_k = P_1 Q_k P_1^T$ as in the proof of Proposition 4.2. Then, $\mathbf{w}(\widetilde{Q}_k) = P_1 \mathbf{w}(Q_k) = (w_2(Q_k) \;\; w_1(Q_k) \;\; w_3(Q_k) \;\; w_4(Q_k) \;\; \cdots \;\; w_k(Q_k))^T$ such that

$$w_1(\widetilde{Q}_k) > w_2(\widetilde{Q}_k) \geq w_3(\widetilde{Q}_k) > w_4(\widetilde{Q}_k) = \cdots = w_k(\widetilde{Q}_k).$$

(ii) If $w_1(Q_k) < w_3(Q_k)$, we employ $\widehat{Q}_k = P_2 Q_k P_2^T$ as in the proof of Proposition 4.3. Then, $\mathbf{w}(\widehat{Q}_k) = P_2 \mathbf{w}(Q_k) = (w_2(Q_k) \;\; w_3(Q_k) \;\; w_1(Q_k) \;\; w_4(Q_k) \;\; \cdots \;\; w_k(Q_k))^T$ such that

$$w_1(\widehat{Q}_k) > w_2(\widehat{Q}_k) > w_3(\widehat{Q}_k) > w_4(\widehat{Q}_k) = \cdots = w_k(\widehat{Q}_k).$$

Note that for both of the aforementioned cases, we have $\mu = c_1$, $\nu = \max_{2 \leq j \leq k}\{c_j\}$, $\zeta = \eta = 0$, and $b = C > 1$, $q = \frac{1}{C}$. The assumptions of Theorems 2.4 and 3.1 are satisfied whether

$$w_1(\widetilde{Q}_k) > \gamma \quad \text{and} \quad w_k(\widetilde{Q}_k) > \delta = 0, \quad \text{or} \quad w_1(\widehat{Q}_k) > \gamma \quad \text{and} \quad w_k(\widehat{Q}_k) > \delta = 0.$$

Clearly, both inequalities concerning w_k are true, so we need to check for w_1. In this setting,

$$\gamma = (1 - C)\alpha - C(\nu - c_1)^3 < C + (c_1^2 + c_1 + c_2)(C - 1) \quad \Leftrightarrow$$
$$(\nu - c_1)^3 > -1 - (c_1^2 + c_1 + c_2 + \alpha)\left(1 - \frac{1}{C}\right) \quad \Leftrightarrow \frac{(\nu - c_1)^3 + 1}{c_1^2 + c_1 + c_2 + \alpha} + 1 > \frac{1}{C},$$

which is true by hypothesis. Thus, the desired bounds for $\rho(Q_k) = \rho(\widetilde{Q}_k) = \rho(\widehat{Q}_k)$ are derived by applying Theorems 2.4 and 3.1. □

Note that if we apply Lemma 2.3, we may obtain the next bounds for the spectral radius of the generalized k-Fibonacci matrix

$$1 \leq \rho(Q_k) \leq \sqrt[3]{C + (c_1^2 + c_1 + c_2)(C - 1)}. \tag{4.21}$$

In the following example, we make a comparison among all bounds discussed in this section for the spectral radius of a generalized k-Fibonacci matrix.

Example 4.5. Let the generalized 4-Fibonacci matrix $Q_4 = \begin{pmatrix} 12 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ with spectral radius $\rho(Q_4) = 12.0901$ and $C = 15$, $\nu = 1$. Obviously, the assumption of Proposition 4.4 is ensured, and then we obtain $w_1(Q_4) = 2{,}213$, $w_2(Q_4) = 1783.67$, $w_3(Q_4) = 183$, and $w_4(Q_4) = 15$. In a addition, $\alpha = 1{,}842$ and $\gamma = -5{,}823$. Then, $Z_1 = w_1(Q_4) = 2{,}213$, $Z_2 = 2194.14$, $Z_3 = 3250.17$, and $Z_4 = 3437.35$.

In Table 3, we list the lower and upper bounds for $\rho(Q_4)$ obtained by applying the formulae presented in this section. We observe that the upper bound obtained by Proposition 4.4 (our proposed formula) is the

Table 3: Comparison among different (lower and upper) bounds for the spectral radius

Bounds for $\rho(Q_4)$	Lower value	Upper value
Proposition 4.1	1	12.2450
Proposition 4.2	1	14.5880
Proposition 4.3	1	13.4245
Proposition 4.4	2.4662	12.9944
Relation (4.21)	1	13.0315
Spectral radius		12.0901

second nearest estimate to $\rho(Q_4)$ and the lower bound obtained by the same Proposition, although far from $\rho(Q_4)$, it has been improved to a certain degree compared to all the other lower bounds that are equal to 1.

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