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# Some inequalities on the spectral radius of matrices

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**Abstract**

Let $A_1, A_2, \ldots, A_k$ be nonnegative matrices. In this paper, some upper bounds for the spectral radius $\rho(A_1 \circ A_2 \circ \cdots \circ A_k)$ are proposed. These bounds generalize some existing results, and comparisons between these bounds are also considered.

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**Keywords:** spectral radius; nonnegative matrix; Hadamard product

## 1 Introduction

Let $M_n$ denote the set of all $n \times n$ complex matrices and $A = (a_{ij}), B = (b_{ij}) \in M_n$. If $a_{ij} - b_{ij} \geq 0$, we say that $A \geq B$, and if $a_{ij} \geq 0$, we say that $A$ is nonnegative, denoted by $A \geq 0$. The symbol $\rho(A)$ stands for the spectral radius of $A$. If $A$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of $A$.

If there does not exist a permutation matrix $P$ such that

$$P^T A P = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where $A_1$, $A_2$ are square matrices, then $A$ is called irreducible.

Let $A$ be an irreducible nonnegative matrix. It is well known that there exists a positive vector $u$ such that $Au = \rho(A)u$, $u$ being called a right Perron eigenvector of $A$.

The Hadamard product of $A$, $B$ is defined as $A \circ B = (a_{ij}b_{ij}) \in M_n$.

Let $A \geq 0$, $B \geq 0$. By using the Gersgorin theorem, Brauer theorem and Brualdi theorem, respectively, the authors of [1–5] have given some inequalities for the upper bounds of $\rho(A \circ B)$. Audenaert [6], Horn and Zhang [7] proved a beautiful inequality on $\rho(A \circ B)$ for nonnegative matrices $A$ and $B$, that is, $\rho(A \circ B) \leq \rho(AB)$. Huang [8] generalized the above inequality to any $k$ nonnegative matrices, that is, $\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leq \rho(A_1 A_2 \cdots A_k)$. Motivated by [8] and [1–4, 9, 10], in this paper we propose some inequalities on the upper bounds for the spectral radius of the Hadamard product of any $k$ nonnegative matrices. These bounds generalize some existing results, and some comparisons between these bounds are also considered.

## 2 Main results

First, we give some lemmas which are useful for obtaining the main results.

**Lemma 2.1** ([11]) *Let $A \in M_n$ be a nonnegative matrix. If $A_k$ is a principal submatrix of $A$, then $\rho(A_k) \le \rho(A)$. If $A$ is irreducible and $A_k \ne A$, then $\rho(A_k) < \rho(A)$.*

**Lemma 2.2** ([11]) *If $A \in M_n$ is an irreducible nonnegative matrix, and $Az \le kz$ for a nonzero nonnegative vector $z$, then $\rho(A) \le k$.*

**Lemma 2.3** ([12]) *Let $A = (a_{ij}) \in M_n$ be a nonnegative matrix. Then*

$$\rho(A) \le \max_{i \ne j} \frac{1}{2}\left\{a_{ii} + a_{jj} + \left[(a_{ii} - a_{jj})^2 + 4\sum_{k \ne i} a_{ik} \sum_{k \ne j} a_{jk}\right]^{\frac{1}{2}}\right\}.$$

**Lemma 2.4** *Let $A_1, A_2, \dots, A_k \in M_n$ and $D_1, D_2, \dots, D_k$ be diagonal matrices of order $n$, then*

$$\begin{aligned} & D^{-1}(A_1 \circ A_2 \circ \cdots \circ A_k)D \\ & \quad = \left(D_1^{-1}A_1D_1\right) \circ \left(D_2^{-1}A_2D_2\right) \circ \cdots \circ \left(D_k^{-1}A_kD_k\right), \end{aligned}$$

*where $D$ equals the product of the matrices $D_k, D_{k-1}, \dots, D_1$, that is, $D = D_k \cdots D_2D_1$.*

*Proof* Let $d_{r,i}$ be the $i$th diagonal of $D_r$ and $a_{r,ij}$ be the $(i,j)$ entry of $A_r$ ($r = 1, 2, \dots, k$). Then the $(i,j)$ entry of $D^{-1}(A_1 \circ A_2 \circ \cdots \circ A_k)D$ is

$$\frac{1}{\prod_{r=1}^k d_{r,i}} \left(\prod_{r=1}^k a_{r,ij}\right) \prod_{r=1}^k d_{r,j} = \prod_{r=1}^k \left(\frac{1}{d_{r,i}} a_{r,ij} d_{r,j}\right),$$

which coincides with the $(i,j)$ entry of $(D_1^{-1}A_1D_1) \circ (D_2^{-1}A_2D_2) \circ \cdots \circ (D_k^{-1}A_kD_k)$. The proof is completed. □

**Theorem 2.1** *Let $A_1, A_2, \dots, A_k \in M_n$ and $A_1 = (a_{ij}) \ge 0, A_2 = (b_{ij}) \ge 0, \dots, A_k = (k_{ij}) \ge 0$. Then*

$$\begin{aligned} & \rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\ & \quad \le \max_{1 \le i \le n} \left\{a_{ii}b_{ii} \cdots k_{ii} + \left(\rho(A_1) - a_{ii}\right)\left(\rho(A_2) - b_{ii}\right) \cdots \left(\rho(A_k) - k_{ii}\right)\right\}. \end{aligned} \tag{2.1}$$

*Proof* If $A_1 \circ A_2 \circ \cdots \circ A_k$ is irreducible, then $A_1, A_2, \dots, A_k$ are all irreducible. From Lemma 2.1, we have

$$\rho(A_1) - a_{ii} > 0, \qquad \rho(A_2) - b_{ii} > 0, \qquad \dots, \qquad \rho(A_k) - k_{ii} > 0, \quad \forall i \in N.$$

Since $A_1, A_2, \dots, A_k$ are nonnegative irreducible, there exist $k$ positive vectors $u, v, \dots, w$ such that $A_1u = \rho(A_1)u, A_2^Tv = \rho(A_2)v, \dots, A_k^Tw = \rho(A_k)w$. Thus, we have

$$a_{ii}u_i + \sum_{j \ne i} a_{ij}u_j = \rho(A_1)u_i, \quad \forall i \in N,$$

$$b_{jj}v_j + \sum_{i \ne j} b_{ij}v_i = \rho(A_2)v_j, \quad \forall j \in N,$$

$$\ldots,$$

$$k_{jj}w_j + \sum_{i\neq j} k_{ij}w_i = \rho(A_k)w_j, \quad \forall j \in N.$$

Thus, we have

$$b_{ij} \leq \frac{[\rho(A_2) - b_{jj}]v_j}{v_i}, \qquad \ldots, \qquad k_{ij} \leq \frac{[\rho(A_k) - k_{jj}]w_j}{w_i}.$$

Let $z$ be the vector $(z_i)$, where

$$z_i = \frac{u_i}{(\rho(A_2) - b_{ii})v_i \cdots (\rho(A_k) - k_{ii})w_i} > 0, \quad \forall i \in N.$$

We define $P = A_1 \circ A_2 \circ \cdots \circ A_k$. For any $i \in N$,

$$\begin{aligned}(Pz)_i &= a_{ii}b_{ii}\cdots k_{ii}z_i + \sum_{i\neq j} a_{ij}b_{ij}\cdots k_{ij}z_j \\ &\leq a_{ii}b_{ii}\cdots k_{ii}z_i + \sum_{i\neq j} a_{ij}\frac{(\rho(A_2) - b_{jj})v_j}{v_i}\cdots\frac{(\rho(A_k) - k_{jj})w_j}{w_i}z_j.\end{aligned}$$

For

$$z_j = \frac{u_j}{(\rho(A_2) - b_{jj})v_j \cdots (\rho(A_k) - k_{jj})w_j},$$

we have

$$\begin{aligned}(Pz)_i &\leq a_{ii}b_{ii}\cdots k_{ii}z_i + \frac{1}{v_i\cdots w_i}\sum_{i\neq j} a_{ij}u_j \\ &= a_{ii}b_{ii}\cdots k_{ii}z_i + \frac{1}{v_i\cdots w_i}\bigl(\rho(A_1) - a_{ii}\bigr)u_i \\ &= a_{ii}b_{ii}\cdots k_{ii}z_i + \bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigl(\rho(A_1) - a_{ii}\bigr)z_i \\ &= \bigl\{a_{ii}b_{ii}\cdots k_{ii} + \bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigr\}z_i.\end{aligned}$$

By Lemma 2.2, this shows that

$$\rho(A_1 \circ \cdots \circ A_k) \leq \max_{1\leq i\leq n}\bigl\{a_{ii}b_{ii}\cdots k_{ii} + \bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigr\}.$$

If $A_1 \circ A_2 \circ \cdots \circ A_k$ is reducible, we denote by $P = (p_{ij})$ the $n \times n$ permutation matrix with $p_{12} = p_{23} = \cdots = p_{n1} = 1$, the remaining $p_{ij} = 0$, then all $A_1 + tP, A_2 + tP, \ldots, A_k + tP$ are nonnegative irreducible matrices for any chosen positive real numbers $t$. We substitute $A_1 + tP, A_2 + tP, \ldots, A_k + tP$ for $A_1, A_2, \ldots, A_k$, respectively, in the previous case, and then, letting $t \to 0$, the result follows by continuity. The proof is completed. □

Setting $k = 2$ in Theorem 2.1, we have the following corollary.

**Corollary 2.1** ([1]) *Let $A_1, A_2 \in M_n$ and $A_1 \geq 0, A_2 \geq 0$. Then*

$$\rho(A_1 \circ A_2) \leq \max_{1 \leq i \leq n} \bigl\{a_{ii}b_{ii} + \bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\bigr\}.$$

**Theorem 2.2** *Let $A_1, A_2, \ldots, A_k \in M_n$ and $A_1 = (a_{ij}) \geq 0, A_2 = (b_{ij}) \geq 0, \ldots, A_k = (k_{ij}) \geq 0$. Then*

$$\begin{aligned} &\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\ &\quad \leq \max_{i \neq j} \frac{1}{2} \bigl\{a_{ii}b_{ii} \cdots k_{ii} + a_{jj}b_{jj} \cdots k_{jj} + \bigl[(a_{ii}b_{ii} \cdots k_{ii} - a_{jj}b_{jj} \cdots k_{jj})^2 \\ &\qquad + 4\bigl(\rho(A_1) - a_{ii}\bigr) \cdots \bigl(\rho(A_k) - k_{ii}\bigr)\bigl(\rho(A_1) - a_{jj}\bigr) \cdots \bigl(\rho(A_k) - k_{jj}\bigr)\bigr]^{\frac{1}{2}}\bigr\}. \end{aligned} \tag{2.2}$$

*Proof* First we assume that $A_1 \circ A_2 \circ \cdots \circ A_k$ is irreducible. Obviously, $A_1, A_2, \ldots, A_k$ are all irreducible, from Lemma 2.1, we have

$$\rho(A_1) - a_{ii} > 0, \qquad \rho(A_2) - b_{ii} > 0, \qquad \ldots, \qquad \rho(A_k) - k_{ii} > 0, \quad \forall i \in N.$$

For the irreducibility of $A_1, A_2, \ldots, A_k$, there exist $k$ positive vectors $u = (u_i), v = (v_i), \ldots, w = (w_i)$ such that $A_1 u = \rho(A_1)u, A_2 v = \rho(A_2)v, \ldots, A_k w = \rho(A_k)w$. Thus, we have

$$\begin{aligned} &a_{ii} + \sum_{j \neq i} \frac{a_{ij}u_j}{u_i} = \rho(A_1), \quad \forall i \in N, \\ &b_{ii} + \sum_{j \neq i} \frac{b_{ij}v_j}{v_i} = \rho(A_2), \quad \forall i \in N, \\ &\ldots, \\ &k_{ii} + \sum_{j \neq i} \frac{k_{ij}w_j}{w_i} = \rho(A_k), \quad \forall i \in N. \end{aligned}$$

Define

$$\begin{aligned} &U = \operatorname{diag}(u_1, u_2, \ldots, u_n), \qquad V = \operatorname{diag}(v_1, v_2, \ldots, v_n), \qquad \ldots, \\ &W = \operatorname{diag}(w_1, w_2, \ldots, w_n). \end{aligned}$$

Let

$$\begin{aligned} &\hat{A}_1 = (\hat{a}_{ij}) = U^{-1}A_1U = \left(\frac{1}{u_i}a_{ij}u_j\right), \\ &\hat{A}_2 = (\hat{b}_{ij}) = V^{-1}A_2V\left(\frac{1}{v_i}b_{ij}v_j\right), \\ &\ldots, \\ &\hat{A}_k = (\hat{k}_{ij}) = W^{-1}A_kW\left(\frac{1}{w_i}k_{ij}w_j\right). \end{aligned}$$

It is easy to show that $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k$ are all nonnegative irreducible matrices, and all the row sums of $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k$ are equal to $\rho(A_1), \rho(A_2), \ldots, \rho(A_k)$, respectively.

Let $D = W \cdots VU$ be the product of $k$ nonsingular diagonal matrices $U, V, \ldots, W$. According to Lemma 2.4, we have

$$\begin{aligned} &D^{-1}(A_1 \circ A_2 \circ \cdots \circ A_k)D \\ &\quad = \bigl(U^{-1}A_1U\bigr) \circ \bigl(V^{-1}A_2V\bigr) \circ \cdots \circ \bigl(W^{-1}A_kW\bigr) \\ &\quad = \hat{A}_1 \circ \hat{A}_2 \circ \cdots \circ \hat{A}_k. \end{aligned}$$

Thus, we have $\rho(A_1 \circ A_2 \circ \cdots \circ A_k) = \rho(\hat{A}_1 \circ \hat{A}_2 \circ \cdots \circ \hat{A}_k)$. From Lemma 2.3, we have

$$\begin{aligned} &\rho(\hat{A}_1 \circ \hat{A}_2 \circ \cdots \circ \hat{A}_k) \\ &\quad \le \max_{i \neq j} \frac{1}{2} \Biggl\{ \hat{a}_{ii}\hat{b}_{ii} \cdots \hat{k}_{ii} + \hat{a}_{jj}\hat{b}_{jj} \cdots \hat{k}_{jj} + \Biggl[ (\hat{a}_{ii}\hat{b}_{ii} \cdots \hat{k}_{ii} - \hat{a}_{jj}\hat{b}_{jj} \cdots \hat{k}_{jj})^2 \\ &\qquad + 4\Biggl(\sum_{k \neq i} \hat{a}_{ik}\hat{b}_{ik} \cdots \hat{k}_{ik}\Biggr)\Biggl(\sum_{k \neq j} \hat{a}_{jk}\hat{b}_{jk} \cdots \hat{k}_{jk}\Biggr)\Biggr]^{\frac{1}{2}} \Biggr\} \\ &\quad \le \max_{i \neq j} \frac{1}{2} \Biggl\{ a_{ii}b_{ii} \cdots k_{ii} + a_{jj}b_{jj} \cdots k_{jj} + \Biggl[ (a_{ii}b_{ii} \cdots k_{ii} - a_{jj}b_{jj} \cdots k_{jj})^2 \\ &\qquad + 4\Biggl(\sum_{k \neq i} \frac{a_{ik}u_k}{u_i} \sum_{k \neq i} \frac{b_{ik}v_k}{v_i} \cdots \sum_{k \neq i} \frac{k_{ik}w_k}{w_i}\Biggr)\Biggl(\sum_{k \neq j} \frac{a_{jk}u_k}{u_j} \sum_{k \neq j} \frac{b_{ik}v_k}{v_j} \cdots \sum_{k \neq j} \frac{k_{jk}w_k}{w_j}\Biggr)\Biggr]^{\frac{1}{2}} \Biggr\} \\ &\quad = \max_{i \neq j} \frac{1}{2} \bigl\{ a_{ii}b_{ii} \cdots k_{ii} + a_{jj}b_{jj} \cdots k_{jj} + \bigl[ (a_{ii}b_{ii} \cdots k_{ii} - a_{jj}b_{jj} \cdots k_{jj})^2 \\ &\qquad + 4\bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr) \cdots \bigl(\rho(A_k) - k_{ii}\bigr)\bigl(\rho(A_1) - a_{jj}\bigr) \\ &\qquad \times \bigl(\rho(A_2) - b_{jj}\bigr) \cdots \bigl(\rho(A_k) - k_{jj}\bigr)\bigr]^{\frac{1}{2}} \bigr\}. \end{aligned}$$

If $A_1 \circ A_2 \circ \cdots \circ A_k$ is reducible, the proof is similar to Theorem 2.1. So, the proof is completed. □

Setting $k = 2$ in Theorem 2.2, we have the following corollary.

**Corollary 2.2** ([2]) *Let $A_1, A_2 \in M_n$ and $A_1 \ge 0$, $A_2 \ge 0$. Then*

$$\begin{aligned} \rho(A_1 \circ A_2) \le \max_{i \neq j} \frac{1}{2} \bigl\{ &a_{ii}b_{ii} + a_{jj}b_{jj} + \bigl[ (a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ &+ 4\bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\bigl(\rho(A_1) - a_{jj}\bigr)\bigl(\rho(A_2) - b_{jj}\bigr)\bigr]^{\frac{1}{2}} \bigr\}. \end{aligned}$$

We next give a simple comparison between the upper bound in (2.1) and the upper bound in (2.2). Without loss of generality, for $i \neq j$, assume that

$$\begin{aligned} &a_{ii}b_{ii} \cdots k_{ii} + \bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr) \cdots \bigl(\rho(A_k) - k_{ii}\bigr) \\ &\quad \ge a_{jj}b_{jj} \cdots k_{jj} + \bigl(\rho(A_1) - a_{jj}\bigr)\bigl(\rho(A_2) - b_{jj}\bigr) \cdots \bigl(\rho(A_k) - k_{jj}\bigr). \end{aligned}$$

Let $\gamma = a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj}$. From (2.2), we have

$$\begin{aligned}
& a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \bigl[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 \\
&\qquad + 4\bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigl(\rho(A_1) - a_{jj}\bigr) \\
&\qquad \times \bigl(\rho(A_2) - b_{jj}\bigr)\cdots\bigl(\rho(A_k) - k_{jj}\bigr)\bigr]^{\frac{1}{2}} \\
&\quad \le \gamma + \bigl\{(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 + 4\bigl(\rho(A_1) - a_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr) \\
&\qquad \times \bigl[\bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr) + a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj}\bigr]\bigr\}^{\frac{1}{2}} \\
&\quad = \gamma + \bigl[\bigl(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj} + 2\bigl(\rho(A_1) - a_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigr)^2\bigr]^{\frac{1}{2}} \\
&\quad = 2a_{ii}b_{ii}\cdots k_{ii} + 2\bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\
&\quad \le \max_{i\neq j} \frac{1}{2}\bigl\{a_{ii}\cdots k_{ii} + a_{jj}\cdots k_{jj} + \bigl[(a_{ii}\cdots k_{ii} - a_{jj}\cdots k_{jj})^2 + 4\bigl(\rho(A_1) - a_{ii}\bigr) \\
&\qquad \times \bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigl(\rho(A_1) - a_{jj}\bigr)\bigl(\rho(A_2) - b_{jj}\bigr)\cdots\bigl(\rho(A_k) - k_{jj}\bigr)\bigr]^{\frac{1}{2}}\bigr\} \\
&\quad \le \max_{1\le i\le n} \frac{1}{2}\bigl[2a_{ii}b_{ii}\cdots k_{ii} + 2\bigl(\rho(A_1) - a_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigr] \\
&\quad = \max_{1\le i\le n}\bigl[a_{ii}b_{ii}\cdots k_{ii} + \bigl(\rho(A_1) - a_{ii}\bigr)\bigl(\rho(A_2) - b_{ii}\bigr)\cdots\bigl(\rho(A_k) - k_{ii}\bigr)\bigr].
\end{aligned}$$

Hence, bound (2.2) is better than bound (2.1).

In [8], the author proved that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \le \rho(A_1A_2\cdots A_k). \tag{2.3}$$

At present, we cannot give the comparison between bounds (2.1) and (2.3) or bounds (2.2) and (2.3), but the following numerical example shows that bounds (2.1) and (2.2) are better than (2.3). Next,we give an example: Consider four $4 \times 4$ nonnegative matrices

$$A = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 0 & 0.05 & 1 & 1 \\ 0 & 0 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 4 & 0.5 & 0.5 \\ 1 & 0 & 3 & 0.5 \\ 0.5 & 1 & 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0 & 2 & 0.5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(i) It is easy to calculate that $\rho(A \circ B) = 5.4983$. By inequalities (2.1) and (2.2), we have

$$\rho(A \circ B) \le \max_{1\le i\le 4}\bigl\{a_{ii}b_{ii} + \bigl(\rho(A) - a_{ii}\bigr)\bigl(\rho(B) - b_{ii}\bigr)\bigr\} = 16.3949,$$

and

$$\rho(A \circ B) \leq 11.6478.$$

By inequality (2.3), we have

$$\rho(A \circ B) \leq \rho(AB) = 19.05.$$

(ii) From calculation, we get $\rho(A \circ B \circ C) = 12.0014$. By inequalities (2.1) and (2.2), we have

$$\rho(A \circ B \circ C) \leq \max_{1 \leq i \leq 4} \bigl\{a_{ii}b_{ii}c_{ii} + \bigl(\rho(A) - a_{ii}\bigr)\bigl(\rho(B) - b_{ii}\bigr)\bigl(\rho(C) - c_{ii}\bigr)\bigr\} = 20.8846,$$

and

$$\rho(A \circ B \circ C) \leq 17.8268.$$

By inequality (2.3), we have

$$\rho(A \circ B \circ C) \leq \rho(ABC) = 88.5.$$

(iii) Let $A \circ B \circ C \circ D = G = (g_{ij})$. Then

$$G = \begin{pmatrix} 16 & 0 & 0 & 1 \\ 0 & 0.2 & 0.5 & 0.5 \\ 0 & 0 & 24 & 0.075 \\ 0 & 0.5 & 0 & 16 \end{pmatrix}, \qquad ABCD = \begin{pmatrix} 117.25 & 78.75 & 155.75 & 126 \\ 34.3375 & 23.0625 & 45.6125 & 36.9 \\ 75.375 & 50.625 & 100.125 & 81 \\ 92.125 & 61.875 & 122.375 & 99 \end{pmatrix}.$$

It is easy to calculate that $\rho(G) = 24.0001$. By inequalities (2.1) and (2.2), we have

$$\rho(G) \leq \max_{1 \leq i \leq 4} \bigl\{a_{ii}b_{ii}c_{ii}d_{ii} + \bigl(\rho(A) - a_{ii}\bigr)\bigl(\rho(B) - b_{ii}\bigr)\bigl(\rho(C) - c_{ii}\bigr)\bigl(\rho(D) - d_{ii}\bigr)\bigr\} = 36.6608$$

and

$$\begin{aligned} \rho(G) \leq{} & \max_{i \neq j} \frac{1}{2}\bigl\{g_{ii} + g_{jj} + \bigl[(g_{ii} - g_{jj})^2 + 4\bigl(\rho(A) - a_{ii}\bigr)\bigl(\rho(B) - b_{ii}\bigr)\bigl(\rho(C) - c_{ii}\bigr) \\ & \quad \times \bigl(\rho(D) - d_{ii}\bigr)\bigl(\rho(A) - a_{jj}\bigr)\bigl(\rho(B) - b_{jj}\bigr)\bigl(\rho(C) - c_{jj}\bigr)\bigl(\rho(D) - d_{jj}\bigr)\bigr]^{\frac{1}{2}}\bigr\} \\ ={} & 32.4451. \end{aligned}$$

By inequality (2.3), we have $\rho(G) \leq \rho(ABCD) = 339.44$.

Next, we will give some other inequalities for $\rho(A_1 \circ A_2 \circ \cdots \circ A_k)$. For $A_1 \geq 0$, write $L_1 = A_1 - \operatorname{diag}(a_{11}, \ldots, a_{nn})$. We denote $J_{A_1} = D_1^{-1}L_1$ with $D_1 = \operatorname{diag}(d_{ii})$, where

$$d_{ii} = \begin{cases} a_{ii}, & \text{if } a_{ii} \neq 0, \\ 1, & \text{if } a_{ii} = 0. \end{cases}$$

Then $J_{A_1}$ is nonnegative.

For $A_2 \geq 0$, let $D_2 = \operatorname{diag}(s_{ii}), \ldots,$ for $A_k \geq 0$, let $D_k = \operatorname{diag}(t_{ii})$ with

$$s_{ii} = \begin{cases} b_{ii}, & \text{if } b_{ii} \neq 0, \\ 1, & \text{if } b_{ii} = 0, \end{cases}$$

$\ldots,$

$$t_{ii} = \begin{cases} k_{ii}, & \text{if } k_{ii} \neq 0, \\ 1, & \text{if } k_{ii} = 0, \end{cases}$$

respectively. Then the nonnegative matrix $J_{A_2}, \ldots, J_{A_k}$ can be similarly defined.

**Theorem 2.3** *Let $A_1, A_2, \ldots, A_k \in M_n$ and $A_1 \geq 0, A_2 \geq 0, \ldots, A_k \geq 0$. Then*

$$\begin{aligned} &\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\ &\quad \leq \max_{1 \leq i \leq n} \bigl\{ a_{ii} b_{ii} \cdots k_{ii} + d_{ii} \rho(J_{A_1}) s_{ii} \rho(J_{A_2}) \cdots t_{ii} \rho(J_{A_k}) \bigr\}. \end{aligned} \tag{2.4}$$

*Proof* Let $Q = A_1 \circ A_2 \circ \cdots \circ A_k$. First assume that $Q$ is irreducible. Obviously $A_1, A_2, \ldots, A_k$ are all irreducible, and then $J_{A_1}, J_{A_2}, \ldots, J_{A_k}$ are all nonnegative irreducible, so there exist $k$ positive vectors $x, y, \ldots, h$ such that $J_{A_1} x = \rho(J_{A_1}) x, J_{A_2} y = \rho(J_{A_2}) y, \ldots, J_{A_k} h = \rho(J_{A_k}) h$. So, we have

$$\sum_{j \neq i} \frac{a_{ij} x_j}{x_i} = d_{ii} \rho(J_{A_1}), \qquad \sum_{j \neq i} \frac{b_{ij} y_j}{y_i} = s_{ii} \rho(J_{A_2}), \qquad \ldots, \qquad \sum_{j \neq i} \frac{k_{ij} h_j}{h_i} = t_{ii} \rho(J_{A_k}).$$

Now let $z = (z_i)$ be the vector, where $z_i = (x_i y_i \cdots h_i) > 0$ for all $i$. For the irreducible nonnegative matrix $Q$, we have

$$\begin{aligned} (Qz)_i &= a_{ii} b_{ii} \cdots k_{ii} z_i + \sum_{i \neq j} a_{ij} b_{ij} \cdots k_{ij} z_j \\ &\leq a_{ii} b_{ii} \cdots k_{ii} z_i + \Bigl( \sum_{i \neq j} a_{ij} x_j \Bigr) \Bigl( \sum_{i \neq j} b_{ij} y_j \Bigr) \cdots \Bigl( \sum_{i \neq j} k_{ij} h_j \Bigr) \\ &= a_{ii} b_{ii} \cdots k_{ii} z_i + \bigl( d_{ii} x_i \rho(J_{A_1}) \bigr) \bigl( s_{ii} y_i \rho(J_{A_2}) \bigr) \cdots \bigl( t_{ii} h_i \rho(J_{A_k}) \bigr) \\ &= \bigl\{ a_{ii} b_{ii} \cdots k_{ii} + d_{ii} \rho(J_{A_1}) s_{ii} \rho(J_{A_2}) \cdots t_{ii} \rho(J_{A_k}) \bigr\} z_i. \end{aligned}$$

By Lemma 2.2, this shows that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leq \max_{1 \leq i \leq n} \bigl\{ a_{ii} b_{ii} \cdots k_{ii} + d_{ii} \rho(J_{A_1}) s_{ii} \rho(J_{A_2}) \cdots t_{ii} \rho(J_{A_k}) \bigr\}.$$

The proof is completed. □

Setting $k = 2$ in Theorem 2.3, we have the following corollary.

**Corollary 2.3** ([4]) *Let $A_1, A_2 \in M_n$ and $A_1 \geq 0$, $A_2 \geq 0$. Then*

$$\rho(A_1 \circ A_2) \leq \max_{1 \leq i \leq n} \bigl\{ a_{ii} b_{ii} + d_{ii} \rho(J_{A_1}) s_{ii} \rho(J_{A_2}) \bigr\}.$$

**Theorem 2.4** *Let $A_1, A_2, \dots, A_k \in M_n$ and $A_1 \geq 0, A_2 \geq 0, \dots, A_k \geq 0$. Then*

$$\begin{aligned}
&\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\
&\quad \leq \max_{i \neq j} \frac{1}{2}\bigl\{a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \bigl[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 \\
&\qquad + 4(d_{ii}s_{ii}\cdots t_{ii})(d_{jj}s_{jj}\cdots t_{jj})\bigl(\rho^2(J_{A_1})\rho^2(J_{A_2})\cdots\rho^2(J_{A_k})\bigr)\bigr]^{\frac{1}{2}}\bigr\}.
\end{aligned} \tag{2.5}$$

*Proof* First we assume that $A_1 \circ A_2 \circ \cdots \circ A_k$ is irreducible. Obviously, $J_{A_1}, J_{A_2}, \dots, J_{A_k}$ are all nonnegative irreducible, then there exist $k$ positive vectors $x, y, \dots, h$ such that $J_{A_1}x = \rho(J_{A_1})x, J_{A_2}y = \rho(J_{A_2})y, \dots, J_{A_k}h = \rho(J_{A_k})h$. Thus, we have

$$\sum_{j\neq i}\frac{a_{ij}x_j}{x_i} = d_{ii}\rho(J_{A_1}), \qquad \sum_{j\neq i}\frac{b_{ij}y_j}{y_i} = s_{ii}\rho(J_{A_2}), \qquad \dots, \qquad \sum_{j\neq i}\frac{k_{ij}h_j}{h_i} = t_{ii}\rho(J_{A_k}).$$

Define

$$\begin{aligned}
&X = \operatorname{diag}(x_1, x_2, \dots, x_n), \qquad Y = \operatorname{diag}(y_1, y_2, \dots, y_n), \qquad \dots, \\
&H = \operatorname{diag}(h_1, h_2, \dots, h_n).
\end{aligned}$$

Let

$$\tilde{A_1} = (\tilde{a}_{ij}) = X^{-1}A_1X, \qquad \tilde{A_2} = (\tilde{b}_{ij}) = Y^{-1}A_2Y, \qquad \dots, \qquad \tilde{A_k} = (\tilde{k}_{ij}) = H^{-1}A_kH.$$

From Lemma 2.4, we have

$$\begin{aligned}
&\bigl(X^{-1}Y^{-1}\cdots H^{-1}\bigr)(A_1 \circ A_2 \circ \cdots \circ A_k)(H\cdots YX) \\
&\quad = \bigl(X^{-1}A_1X\bigr) \circ \bigl(Y^{-1}A_2Y\bigr) \circ \cdots \circ \bigl(H^{-1}A_kH\bigr) \\
&\quad = \tilde{A_1} \circ \tilde{A_2} \circ \cdots \circ \tilde{A_k}.
\end{aligned}$$

Thus, $\rho(A_1 \circ A_2 \circ \cdots \circ A_k) = \rho(\tilde{A_1} \circ \tilde{A_2} \circ \cdots \circ \tilde{A_k})$. From Lemma 2.3, we have

$$\begin{aligned}
&\rho(\tilde{A_1} \circ \tilde{A_2} \circ \cdots \circ \tilde{A_k}) \\
&\quad \leq \max_{i\neq j}\frac{1}{2}\Biggl\{a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \Biggl[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 \\
&\qquad + 4\Biggl(\sum_{k\neq i}\frac{a_{ik}x_k}{x_i}\frac{b_{ik}y_k}{y_i}\cdots\frac{k_{ik}h_k}{h_i}\Biggr)\Biggl(\sum_{k\neq j}\frac{a_{jk}x_k}{x_j}\frac{b_{ik}y_k}{y_j}\cdots\frac{k_{jk}h_k}{h_j}\Biggr)\Biggr]^{\frac{1}{2}}\Biggr\} \\
&\quad \leq \max_{i\neq j}\frac{1}{2}\Biggl\{a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \Biggl[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 \\
&\qquad + 4\Biggl(\sum_{k\neq i}\frac{a_{ik}x_k}{x_i}\sum_{k\neq i}\frac{b_{ik}y_k}{y_i}\cdots\sum_{k\neq i}\frac{k_{ik}h_k}{h_i}\Biggr)\Biggl(\sum_{k\neq j}\frac{a_{jk}x_k}{x_j}\sum_{k\neq j}\frac{b_{ik}y_k}{y_j}\cdots\sum_{k\neq j}\frac{k_{jk}h_k}{h_j}\Biggr)\Biggr]^{\frac{1}{2}}\Biggr\} \\
&\quad = \max_{i\neq j}\frac{1}{2}\bigl\{a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \bigl[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 \\
&\qquad + 4(d_{ii}s_{ii}\cdots t_{ii})(d_{jj}s_{jj}\cdots t_{jj})\bigl(\rho^2(J_{A_1})\rho^2(J_{A_2})\cdots\rho^2(J_{A_k})\bigr)\bigr]^{\frac{1}{2}}\bigr\}.
\end{aligned}$$

If $A_1 \circ A_2 \circ \cdots \circ A_k$ is reducible, then substituting $A_1 + tP, A_2 + tP, \ldots, A_k + tP$ for $A_1, A_2, \ldots, A_k$, respectively, in the previous case, letting $t \to 0$, the result is derived. □

Setting $k = 2$ in Theorem 2.4, we have the following corollary.

**Corollary 2.4** ([1]) *Let $A_1, A_2 \in M_n$ and $A_1 \geq 0$, $A_2 \geq 0$. Then*

$$\rho(A_1 \circ A_2) \leq \max_{i \neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} + \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4d_{ii}s_{ii}d_{jj}s_{jj}\rho^2(J_{A_1})\rho^2(J_{A_2})\right]^{\frac{1}{2}}\right\}.$$

We next give a comparison between the upper bound in (2.4) and the upper bound in (2.5). Without loss of generality, for $i \neq j$, assume that

$$\begin{aligned} &a_{ii}b_{ii}\cdots k_{ii} + d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k}) \\ &\quad \geq a_{jj}b_{jj}\cdots k_{jj} + d_{jj}s_{jj}\cdots t_{jj}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k}). \end{aligned}$$

Let $\gamma = a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj}$. From (2.5), we have

$$\begin{aligned} &a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \left[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2\right. \\ &\qquad \left. + 4(d_{ii}s_{ii}\cdots t_{ii})(d_{jj}s_{jj}\cdots t_{jj})\left(\rho^2(J_{A_1})\rho^2(J_{A_2})\cdots\rho^2(J_{A_k})\right)\right]^{\frac{1}{2}} \\ &\quad \leq \gamma + \left\{(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2 + 4d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k})\right. \\ &\qquad \left. \times\left[d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k}) + a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj}\right]\right\}^{\frac{1}{2}} \\ &\quad = \gamma + \left[\left(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj} + 2d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k})\right)^2\right]^{\frac{1}{2}} \\ &\quad = 2a_{ii}b_{ii}\cdots k_{ii} + 2d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k}). \end{aligned}$$

Thus, from (2.5) and the above inequality, we have

$$\begin{aligned} &\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \\ &\quad \leq \max_{i \neq j} \frac{1}{2}\left\{a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} + \left[(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj})^2\right.\right. \\ &\qquad \left.\left. + 4(d_{ii}s_{ii}\cdots t_{ii})(d_{jj}s_{jj}\cdots t_{jj})\left(\rho^2(J_{A_1})\rho^2(J_{A_2})\cdots\rho^2(J_{A_k})\right)\right]^{\frac{1}{2}}\right\} \\ &\quad \leq \max_{1 \leq i \leq n} \frac{1}{2}\left(2a_{ii}b_{ii}\cdots k_{ii} + 2d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k})\right) \\ &\quad = \max_{1 \leq i \leq n}\left(a_{ii}b_{ii}\cdots k_{ii} + d_{ii}s_{ii}\cdots t_{ii}\rho(J_{A_1})\rho(J_{A_2})\cdots\rho(J_{A_k})\right). \end{aligned}$$

Hence, the upper bound (2.5) is better than bound (2.4). Here too, we could not give the comparison between (2.4) and (2.3) or (2.5) and (2.3). Next, we give an example which shows that the results obtained in Theorems 2.3 and 2.4 are better than inequalities (2.3).

Let

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 4 & 0.5 & 0.5 \\ 1 & 0 & 3 & 0.5 \\ 0.5 & 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0 & 2 & 0.5 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0.5 & 2 & 0.5 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 1 & 0.5 & 1 \end{pmatrix}.$$

Let $A \circ B \circ C \circ D = P = (p_{ij})$. Then

$$P = \begin{pmatrix} 16 & 0 & 1 & 0.25 \\ 1 & 8 & 0.25 & 0 \\ 0 & 0 & 12 & 0.075 \\ 0 & 1 & 0.5 & 4 \end{pmatrix}, \qquad ABCD = \begin{pmatrix} 35.5 & 55.75 & 86 & 35.75 \\ 57.75 & 91.875 & 139 & 57.875 \\ 30.25 & 57.25 & 78 & 34.75 \\ 34.875 & 64.125 & 87.5 & 38.875 \end{pmatrix}.$$

It is easy to calculate that $\rho(P) = 16.0028$. By inequalities (2.4) and (2.5), we have

$$\rho(P) \leq \max_{1 \leq i \leq 4} \bigl\{ p_{ii} \bigl( 1 + \rho(J_A)\rho(J_B)\rho(J_C)\rho(J_D) \bigr) \bigr\} = 36.2262$$

and

$$\begin{aligned} \rho(P) &\leq \max_{i \neq j} \frac{1}{2} \bigl\{ p_{ii} + p_{jj} + \bigl[ (p_{ii} - p_{jj})^2 + 4 p_{ii} p_{jj} \rho^2(J_A)\rho^2(J_B)\rho^2(J_C)\rho^2(J_D) \bigr]^{\frac{1}{2}} \bigr\} \\ &= 29.6605. \end{aligned}$$

By inequality (2.3) and Lemma 2.1, we have $\rho(P) \leq 91.875$.

## 3 Conclusions

In this paper, we have proposed some upper bounds for $\rho(A_1 \circ A_2 \circ \cdots \circ A_k)$ of the Hadamard product of matrices. These bounds generalize some corresponding results of [1–4].

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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