Linear Algebra and its Applications 449 (2014) 194–209

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Sharp bounds for the spectral radius of nonnegative matrices

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ARTICLE INFO

Article history:
Received 21 November 2013
Accepted 17 February 2014
Available online 12 March 2014
Submitted by X. Zhan

MSC:
15A18
05C50

Keywords:
Nonnegative matrix
Spectral radius
Average 2-row sum
Adjacency matrix
Signless Laplacian matrix
Distance-based matrix

ABSTRACT

We give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with all row sums positive using its average 2-row sums, and characterize the equality cases if the matrix is irreducible. We compare these bounds with the known bounds using the row sums by examples. We also apply these bounds to various matrices associated with a graph, including the adjacency matrix, the signless Laplacian matrix and some distance-based matrices. Some known results are generalized and improved.

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http://dx.doi.org/10.1016/j.laa.2014.02.031

1. Introduction

Let A be an $n \times n$ nonnegative matrix. The spectral radius (alias Perron root) of A, denoted by $\rho(A)$, is the largest modulus of eigenvalues of A. See [2,8,13,16,18,21,22] for some known properties of the spectral radius of nonnegative matrices.

In this paper, we also consider the spectral radius of some nonnegative matrices associated with a graph. Let G be a simple undirected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$.

The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise [5]. For $1 \leqslant i \leqslant n$, let d_i be the degree of vertex v_i in G. Let $Deg(G)$ be the degree diagonal matrix $\operatorname{diag}(d_1, \ldots, d_n)$. The signless Laplacian matrix of G is the $n \times n$ matrix $Q(G) = Deg(G) + A(G)$ [7]. The spectral radius of the adjacency matrix has been studied extensively (see, e.g., [6,8,12,15,19]), and the spectral radius of the signless Laplacian matrix has also received much attention (see, e.g., [8,11,20,23]).

Suppose that G is connected. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_{ij})$, where d_{ij} is the distance between vertices v_i and v_j, i.e., the number of edges of a shortest path connecting them, in G [9,14]. For $1 \leqslant i \leqslant n$, the transmission D_i of vertex v_i in G is the sum of distances between v_i and (other) vertices of G. Let $Tr(G)$ be the transmission diagonal matrix $\operatorname{diag}(D_1, \ldots, D_n)$. The distance signless Laplacian matrix of G is the $n \times n$ matrix $\mathcal{Q}(G) = Tr(G) + D(G)$ [1]. The reciprocal distance matrix (alias Harary matrix) of G is the $n \times n$ matrix $R(G) = (r_{ij})$, where $r_{ij} = \frac{1}{d_{ij}}$ for $i \neq j$, and $r_{ii} = 0$ for $1 \leqslant i \leqslant n$ [14]. Some results have been obtained for the spectral radius of these distance-based matrices of a connected graph (see, e.g., [8,24]).

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. For $1 \leqslant i \leqslant n$, the i-th row sum of A is $r_i(A) = \sum_{j=1}^{n} a_{ij}$. Duan and Zhou [8] found upper and lower bounds for the spectral radius of a nonnegative matrix using its row sums, and characterized the equality cases if the matrix is irreducible. They also applied those bounds to the nonnegative matrices associated with a graph as mentioned above.

For $1 \leqslant i \leqslant n$ and an $n \times n$ nonnegative matrix $A = (a_{ij})$ with $r_i(A) > 0$, the i-th average 2-row sum of A is defined as $m_i(A) = \frac{\sum_{k=1}^{n} a_{ik} r_k(A)}{r_i(A)}$. For a graph G on n vertices with $d_i > 0$, $m_i(A(G)) = \frac{\sum_{v_i v_j \in E(G)} d_j}{d_i}$, which is known as the average 2-degree of vertex v_i in G [3,17]. Huang and Weng [10] gave an upper bound for the spectral radius of the adjacency matrix of a connected graph with at least two vertices using its average 2-degrees (cf. Chen et al. [4]).

In this paper, we give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with all row sums positive using the average 2-row sums, and characterize the equality cases if the matrix is irreducible. Then we compare these bounds with those using the row sums presented in [8] by examples. We also apply these results to various matrices associated with a graph as mentioned above. Some known results are generalized and improved.

2. Bounds for the spectral radius of nonnegative matrices

The following lemma is well known.

Lemma 2.1. *(See [18, p. 24].) If A is an $n \times n$ nonnegative matrix, then*

$$\min_{1 \leqslant i \leqslant n} r_i(A) \leqslant \rho(A) \leqslant \max_{1 \leqslant i \leqslant n} r_i(A).$$

Moreover, if A is irreducible, then either equality holds if and only if $r_1(A) = \cdots = r_n(A)$.

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with all row sums positive. Let $U = \operatorname{diag}(r_1(A), \ldots, r_n(A))$ and $B = (b_{ij}) = U^{-1}AU$. Obviously, $b_{ij} = \frac{a_{ij} r_j(A)}{r_i(A)}$ for $1 \leqslant i, j \leqslant n$. Thus $r_i(B) = \sum_{k=1}^{n} b_{ik} = \frac{\sum_{k=1}^{n} a_{ik} r_k(A)}{r_i(A)} = m_i(A)$ for $1 \leqslant i \leqslant n$. By Lemma 2.1, we have the following lemma.

Lemma 2.2. *(See [18, pp. 27–28].) Let A be an $n \times n$ nonnegative matrix with all row sums positive. Then*

$$\min_{1 \leqslant i \leqslant n} m_i(A) \leqslant \rho(A) \leqslant \max_{1 \leqslant i \leqslant n} m_i(A).$$

Moreover, if A is irreducible, then either equality holds if and only if $m_1(A) = \cdots = m_n(A)$.

Theorem 2.1. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with all row sums positive and with average 2-row sums $m_1 \geqslant \cdots \geqslant m_n$. Let M be the largest diagonal element, and N the largest off-diagonal element of A. Suppose that $N > 0$. Let $b = \max\{\frac{r_j(A)}{r_i(A)}: 1 \leqslant i, j \leqslant n\}$. For $1 \leqslant l \leqslant n$, let*

$$\phi_l = \frac{m_l + M - Nb + \sqrt{(m_l - M + Nb)^2 + 4Nb \sum_{i=1}^{l-1}(m_i - m_l)}}{2}.$$

Then $\rho(A) \leqslant \phi_l$ for $1 \leqslant l \leqslant n$. Moreover, if A is irreducible, then $\rho(A) = \phi_l$ for some l with $1 \leqslant l \leqslant n$ if and only if $m_1 = \cdots = m_n$, or for some t with $2 \leqslant t \leqslant l$, A satisfies the following conditions:

(i) $a_{ii} = M$ *for* $1 \leqslant i \leqslant t-1$,
(ii) $a_{ik} = N$ *and* $\frac{r_k(A)}{r_i(A)} = b$ *for* $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant t-1$ *and* $k \neq i$,
(iii) $m_t = \cdots = m_n$.

Proof. Let $r_i = r_i(A)$ for $1 \leqslant i \leqslant n$. Since $m_i \geqslant a_{ii}$ for $1 \leqslant i \leqslant n$, we have $m_1 \geqslant M$.

If $l = 1$, then $\phi_l = \frac{m_1 + M - Nb + |m_1 - M + Nb|}{2} = m_1$, and thus the result follows immediately from Lemma 2.2.

Suppose in the following that $2 \leqslant l \leqslant n$.

Let $U = \operatorname{diag}(r_1x_1, \dots, r_{l-1}x_{l-1}, r_l, \dots, r_n)$, where $x_i \geqslant 1$ is a variable to be determined later for $1 \leqslant i \leqslant l-1$. Let $B = U^{-1}AU$. Obviously, A and B have the same eigenvalues.

For $1 \leqslant i \leqslant l-1$, since $a_{ii} \leqslant M$, and $a_{ik} \leqslant N$, $\frac{r_k}{r_i} \leqslant b$ for $k \neq i$, we have

$$\begin{aligned} r_i(B) &= \frac{1}{x_i}\left(\sum_{k=1}^{l-1} a_{ik}\frac{r_k}{r_i}x_k + \sum_{k=l}^{n} a_{ik}\frac{r_k}{r_i}\right) \\ &= \frac{1}{x_i}\left(a_{ii}(x_i-1) + \sum_{\substack{1\leqslant k\leqslant l-1\\ k\neq i}} a_{ik}\frac{r_k}{r_i}(x_k-1) + \sum_{k=1}^{n} a_{ik}\frac{r_k}{r_i}\right) \\ &\leqslant \frac{1}{x_i}\left(M(x_i-1) + Nb\sum_{\substack{1\leqslant k\leqslant l-1\\ k\neq i}}(x_k-1) + m_i\right) \end{aligned}$$

with equality if and only if (a) and (b) hold: (a) $x_i = 1$ or $a_{ii} = M$, (b) $x_k = 1$, or $a_{ik} = N$ and $\frac{r_k}{r_i} = b$, where $1 \leqslant k \leqslant l-1$ and $k \neq i$.

For $l \leqslant i \leqslant n$, since $m_i \leqslant m_l$, and $a_{ik} \leqslant N$, $\frac{r_k}{r_i} \leqslant b$ for $1 \leqslant k \leqslant l-1$, we have

$$\begin{aligned} r_i(B) &= \sum_{k=1}^{l-1} a_{ik}\frac{r_k}{r_i}x_k + \sum_{k=l}^{n} a_{ik}\frac{r_k}{r_i} \\ &= \sum_{k=1}^{l-1} a_{ik}\frac{r_k}{r_i}(x_k-1) + \sum_{k=1}^{n} a_{ik}\frac{r_k}{r_i} \\ &= \sum_{k=1}^{l-1} a_{ik}\frac{r_k}{r_i}(x_k-1) + m_i \\ &\leqslant Nb\sum_{k=1}^{l-1}(x_k-1) + m_l \end{aligned}$$

with equality if and only if (c) and (d) hold: (c) $x_k = 1$, or $a_{ik} = N$ and $\frac{r_k}{r_i} = b$, where $1 \leqslant k \leqslant l-1$, (d) $m_i = m_l$.

From the definition of ϕ_l with $1 \leqslant l \leqslant n$, we have $\phi_l^2 - (m_l + M - Nb)\phi_l + m_l(M - Nb) - Nb\sum_{k=1}^{l-1}(m_k - m_l) = 0$, i.e., $Nb\sum_{k=1}^{l-1}(m_k - m_l) = (\phi_l - m_l)(\phi_l - M + Nb)$. Note that $Nb > 0$. If $\sum_{k=1}^{l-1}(m_k - m_l) > 0$, then $\phi_l > \frac{m_l+M-Nb+|m_l-M+Nb|}{2} \geqslant \frac{m_l+M-Nb-(m_l-M+Nb)}{2} = M - Nb$, and if $m_1 = \cdots = m_l$, then since $m_1 \geqslant M$, we have $\phi_l = \frac{m_1+M-Nb+|m_1-M+Nb|}{2} > \frac{m_1+M-Nb-(m_1-M+Nb)}{2} = M - Nb$. Thus $\phi_l - M + Nb > 0$. For $1 \leqslant i \leqslant l-1$, let $x_i = 1 + \frac{m_i - m_l}{\phi_l - M + Nb}$. Obviously, $x_i \geqslant 1$ and

$$Nb\sum_{k=1}^{l-1}(x_k-1) = \frac{Nb\sum_{k=1}^{l-1}(m_k - m_l)}{\phi_l - M + Nb} = \phi_l - m_l.$$

Thus, for $1 \leqslant i \leqslant l-1$,

$$\begin{aligned} r_i(B) &\leqslant \frac{1}{x_i}\left(Nb\sum_{k=1}^{l-1}(x_k-1)+(M-Nb)(x_i-1)+m_i\right) \\ &= \frac{(\phi_l-m_l)+(M-Nb)\cdot\frac{m_i-m_l}{\phi_l-M+Nb}+m_i}{1+\frac{m_i-m_l}{\phi_l-M+Nb}} \\ &= \phi_l, \end{aligned}$$

and for $l \leqslant i \leqslant n$,

$$r_i(B) \leqslant Nb\sum_{k=1}^{l-1}(x_k-1)+m_l = (\phi_l-m_l)+m_l = \phi_l.$$

By Lemma 2.1, $\rho(A)=\rho(B)\leqslant \max_{1\leqslant i\leqslant n} r_i(B)\leqslant \phi_l$.

Suppose that A is irreducible. Then B is also irreducible.

Suppose that $\rho(A)=\phi_l$ for some l with $2\leqslant l\leqslant n$. Then $\rho(B)=\max_{1\leqslant i\leqslant n} r_i(B)=\phi_l$. By Lemma 2.1, $r_1(B)=\cdots=r_n(B)=\phi_l$, and thus from the above arguments, (a) and (b) hold for $1\leqslant i\leqslant l-1$, and (c) and (d) hold for $l\leqslant i\leqslant n$. If $m_1=m_l$, then we have from (d) that $m_1=\cdots=m_n$. Suppose that $m_1>m_l$. Let t be the smallest integer such that $m_t=m_l$, where $2\leqslant t\leqslant l$. For $1\leqslant i\leqslant t-1$, since $m_i>m_l$, we have $x_i>1$. Now (i) and (ii) follow from (a), (b) for $1\leqslant i\leqslant l-1$ and (c) for $l\leqslant i\leqslant n$. From (d), we have $m_t=\cdots=m_l=\cdots=m_n$, and thus (iii) holds.

Conversely, if $m_1=\cdots=m_n$, then $\phi_l=m_1$ and by Lemma 2.2, $\rho(A)=m_1$, and thus $\rho(A)=\phi_l$. If (i)–(iii) hold for some l and t with $2\leqslant t\leqslant l\leqslant n$, then (a) and (b) hold for $1\leqslant i\leqslant l-1$, and (c) and (d) hold for $l\leqslant i\leqslant n$, implying that $r_i(B)=\phi_l$ for $1\leqslant i\leqslant n$, and thus by Lemma 2.1, $\rho(A)=\rho(B)=\phi_l$. $\square$

Let I_n and J_n be the $n\times n$ identity matrix and the $n\times n$ all-one matrix, respectively.

Under the notations and conditions of Theorem 2.1, for $1\leqslant i\leqslant n$, since $a_{ii}\leqslant M$, and $a_{ij}\leqslant N$, $\frac{r_j(A)}{r_i(A)}\leqslant b$ for $j\neq i$, we have

$$\sum_{i=1}^n m_i=\sum_{i=1}^n\left(a_{ii}+\sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}} a_{ij}\frac{r_j(A)}{r_i(A)}\right)\leqslant\sum_{i=1}^n\big(M+Nb(n-1)\big)=n(Nbn+M-Nb)$$

with equality if and only if $a_{ii}=M$ for $1\leqslant i\leqslant n$, and $a_{ij}=N$, $\frac{r_j(A)}{r_i(A)}=b$ for $1\leqslant i,j\leqslant n$ and $i\neq j$, or equivalently, $A=MI_n+(N-M)J_n$, implying that $\phi_1=\cdots=\phi_n$. If $A\neq MI_n+(N-M)J_n$, then $\sum_{i=1}^n m_i<n(Nbn+M-Nb)$.

Proposition 2.1. *Under the notations and conditions of Theorem 2.1 with $A\neq MI_n+(N-M)J_n$, let*

$$l = \min\left\{k\colon \sum_{i=1}^{k} m_i < k(Nbk + M - Nb),\ 1 \leqslant k \leqslant n\right\}.$$

Then $\min\{\phi_i\colon 1 \leqslant i \leqslant n\} = \phi_l$.

Proof. Note that $m_1 \geqslant M$. We have $2 \leqslant l \leqslant n$.

From the expression of ϕ_k with $1 \leqslant k \leqslant n$, we have $\phi_k \geqslant \phi_{k+1}$ if and only if

$$\begin{aligned}&(m_k - m_{k+1})\sqrt{(m_k - M + Nb)^2 + 4Nb\sum_{i=1}^{k-1}(m_i - m_k)}\\ &\quad\geqslant (m_k - m_{k+1})(2Nbk + M - Nb - m_k).\end{aligned}$$

Note that

$$\sqrt{(m_k - M + Nb)^2 + 4Nb\sum_{i=1}^{k-1}(m_i - m_k)} \geqslant 2Nbk + M - Nb - m_k$$

if and only if $\sum_{i=1}^{k} m_i \geqslant k(Nbk + M - Nb)$. Thus, if $\sum_{i=1}^{k} m_i \geqslant k(Nbk + M - Nb)$, then $\phi_k \geqslant \phi_{k+1}$, and if $\sum_{i=1}^{k} m_i < k(Nbk + M - Nb)$, then $\phi_k \leqslant \phi_{k+1}$.

For $1 \leqslant k \leqslant l - 1$, by the choice of l, we have $\sum_{i=1}^{k} m_i \geqslant k(Nbk + M - Nb)$, and thus $\phi_1 \geqslant \cdots \geqslant \phi_l$. For $l \leqslant k \leqslant n$, we are to show that $\sum_{i=1}^{k} m_i < k(Nbk+M-Nb)$ by induction on k. The case $k = l$ has been done from the choice of l. Suppose that $\sum_{i=1}^{k} m_i < k(Nbk + M - Nb)$ for some k with $l \leqslant k \leqslant n - 1$. Then $m_k < Nbk + M - Nb$, which, together with the fact that $m_{k+1} \leqslant m_k$, implies that $\sum_{i=1}^{k+1} m_i < k(Nbk + M - Nb) + (Nbk + M - Nb) < (k + 1)(Nb(k + 1) + M - Nb)$. Thus $\sum_{i=1}^{k} m_i < k(Nbk + M - Nb)$ for $l \leqslant k \leqslant n$, and then $\phi_l \leqslant \cdots \leqslant \phi_n$. Thus $\min\{\phi_i\colon 1 \leqslant i \leqslant n\} = \phi_l$. □

Under the notations and conditions of Theorem 2.1, if A is symmetric, then conditions (i)–(iii) hold if and only if conditions (i′) and (ii′) hold:

(i′) $a_{11} = M$ and the off-diagonal elements of A in the first row and column are equal to N,
(ii′) $r_2(A) = \cdots = r_n(A)$.

This is because if $t = 2$, then since (i′) and (ii′) imply that $m_2 = \cdots = m_n$, conditions (i)–(iii) are equivalent to conditions (i′) and (ii′), and if $t \geqslant 3$, then since (i) and (ii) imply that $r_1(A) = \cdots = r_{t-1}(A) = M + N(n - 1)$ and $b = \frac{r_2(A)}{r_1(A)} = 1$, we have $r_1(A) = \cdots = r_n(A)$, and thus conditions (i)–(iii) are equivalent to $A = MI_n + (N - M)J_n$, which also satisfies conditions (i′) and (ii′).

In [8], the following upper bound for the spectral radius was given.

Theorem 2.2. *(See [8].) Let A be an $n \times n$ nonnegative matrix with row sums $r_1 \geqslant \cdots \geqslant r_n$. Let M be the largest diagonal element, and N the largest off-diagonal element of A. Suppose that $N > 0$. For $1 \leqslant l \leqslant n$, let*

$$\Phi_l = \frac{r_l + M - N + \sqrt{(r_l - M + N)^2 + 4N \sum_{i=1}^{l-1}(r_i - r_l)}}{2}.$$

Then $\rho(A) \leqslant \Phi_l$ for $1 \leqslant l \leqslant n$.

Consider

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix}.$$

In notations of Theorem 2.1, $m_1 = 5$, $m_2 = m_3 = m_4 = \frac{23}{5}$, $M = 0$, $N = 2$ and $b = \frac{5}{3}$, implying that $\phi_1 = 5$, $\phi_2 = \phi_3 = \phi_4 = 4.7647$, and thus $\rho(A_1) \leqslant 4.7647$. Obviously, A_1 is permutation similar to

$$A_1' = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and thus $\rho(A_1) = \rho(A_1')$. In notations of Theorem 2.2, for A_1', we have $r_1 = r_2 = r_3 = 5$, $r_4 = 3$, $M = 0$ and $N = 2$, implying that $\Phi_1 = \Phi_2 = \Phi_3 = 5$, $\Phi_4 = 4.7720$, and thus $\rho(A_1) \leqslant 4.7720$. The upper bound in Theorem 2.1 is smaller than that in Theorem 2.2 for A_1.

Now consider

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

In notations of Theorem 2.1, $m_1 = m_2 = m_3 = 3$, $m_4 = 1$, $M = 0$, $N = 1$ and $b = 3$, implying that $\phi_1 = \phi_2 = \phi_3 = 3$, $\phi_4 = 3.6904$, and thus $\rho(A_2) \leqslant 3$. Obviously, A_2 is permutation similar to

$$A_2' = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and thus $\rho(A_2) = \rho(A_2')$. In notations of Theorem 2.2, for A_2', we have $r_1 = 3$, $r_2 = r_3 = r_4 = 1$, $M = 0$ and $N = 1$, implying that $\Phi_1 = 3$, $\Phi_2 = 1.7321$, $\Phi_3 = 2.2361$, $\Phi_4 = 2.6458$, and thus $\rho(A_2) \leqslant 1.732$. The upper bound in Theorem 2.1 is greater than that in Theorem 2.2 for A_2.

Theorem 2.3. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with all row sums positive and with average 2-row sums $m_1 \geqslant \cdots \geqslant m_n$. Let S be the smallest diagonal element, and T the smallest off-diagonal element of A. Let $c = \min\{\frac{r_j(A)}{r_i(A)}\colon 1 \leqslant i,j \leqslant n\}$. Let*

$$\psi_n = \frac{m_n + S - Tc + \sqrt{(m_n - S + Tc)^2 + 4Tc\sum_{i=1}^{n-1}(m_i - m_n)}}{2}.$$

Then $\rho(A) \geqslant \psi_n$. Moreover, if A is irreducible, then $\rho(A) = \psi_n$ if and only if $m_1 = \cdots = m_n$, or $T > 0$ and for some t with $2 \leqslant t \leqslant n$, A satisfies the following conditions:

(i) $a_{ii} = S$ for $1 \leqslant i \leqslant t-1$,
(ii) $a_{ik} = T$ and $\frac{r_k(A)}{r_i(A)} = c$ for $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant t-1$ and $k \neq i$,
(iii) $m_t = \cdots = m_n$.

Proof. Let $r_i = r_i(A)$ for $1 \leqslant i \leqslant n$. Note that $m_n \geqslant a_{nn} \geqslant S$.

If $T = 0$, then $\psi_n = m_n$, and thus the result follows immediately from Lemma 2.2.

Suppose in the following that $T > 0$.

Let $U = \operatorname{diag}(r_1x_1, \ldots, r_{n-1}x_{n-1}, r_n)$, where $x_i \geqslant 1$ is a variable to be determined later for $1 \leqslant i \leqslant n-1$. Let $B = U^{-1}AU$. Obviously, A and B have the same eigenvalues.

For $1 \leqslant i \leqslant n-1$, since $a_{ii} \geqslant S$, and $a_{ik} \geqslant T$, $\frac{r_k}{r_i} \geqslant c$ for $k \neq i$, we have

$$\begin{aligned} r_i(B) &= \frac{1}{x_i}\left(a_{ii}(x_i - 1) + \sum_{\substack{1 \leqslant k \leqslant n-1 \\ k \neq i}} a_{ik}\frac{r_k}{r_i}(x_k - 1) + \sum_{k=1}^{n} a_{ik}\frac{r_k}{r_i}\right) \\ &\geqslant \frac{1}{x_i}\left(S(x_i - 1) + Tc\sum_{\substack{1 \leqslant k \leqslant n-1 \\ k \neq i}}(x_k - 1) + m_i\right) \end{aligned}$$

with equality if and only if (a) and (b) hold: (a) $x_i = 1$ or $a_{ii} = S$, (b) $x_k = 1$, or $a_{ik} = T$ and $\frac{r_k}{r_i} = c$, where $1 \leqslant k \leqslant n-1$ and $k \neq i$.

Similarly,

$$r_n(B) = \sum_{k=1}^{n-1} a_{nk}\frac{r_k}{r_n}(x_k - 1) + \sum_{k=1}^{n} a_{nk}\frac{r_k}{r_n} \geqslant Tc\sum_{k=1}^{n-1}(x_k - 1) + m_n$$

with equality if and only if (c) holds: (c) $x_k = 1$, or $a_{nk} = T$ and $\frac{r_k}{r_n} = c$, where $1 \leqslant k \leqslant n-1$.

From the definition of ψ_n, we have $Tc\sum_{k=1}^{n-1}(m_k - m_n) = (\psi_n - m_n)(\psi_n - S + Tc)$. Note that $Tc > 0$ and $m_n \geqslant S$. If $\sum_{k=1}^{n-1}(m_k - m_n) > 0$, then $\psi_n > \frac{m_n+S-Tc+|m_n-S+Tc|}{2} > \frac{m_n+S-Tc-(m_n-S+Tc)}{2} = S - Tc$, and if $m_1 = \cdots = m_n$, then $\psi_n = m_n \geqslant S > S - Tc$. Thus $\psi_n - S + Tc > 0$. For $1 \leqslant i \leqslant n-1$, let $x_i = 1 + \frac{m_i - m_n}{\psi_n - S + Tc}$. Obviously, $x_i \geqslant 1$ and

$$Tc\sum_{k=1}^{n-1}(x_k - 1) = \frac{Tc\sum_{k=1}^{n-1}(m_k - m_n)}{\psi_n - S + Tc} = \psi_n - m_n.$$

Thus, for $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} r_i(B) &\geqslant \frac{1}{x_i}\left(Tc\sum_{k=1}^{n-1}(x_k - 1) + (S - Tc)(x_i - 1) + m_i\right) \\ &= \frac{(\psi_n - m_n) + (S - Tc)\cdot\frac{m_i - m_n}{\psi_n - S + Tc} + m_i}{1 + \frac{m_i - m_n}{\psi_n - S + Tc}} \\ &= \psi_n, \end{aligned}$$

and

$$r_n(B) \geqslant Tc\sum_{k=1}^{n-1}(x_k - 1) + m_n = (\psi_n - m_n) + m_n = \psi_n.$$

By Lemma 2.1, $\rho(A) = \rho(B) \geqslant \min_{1\leqslant i\leqslant n} r_i(B) \geqslant \psi_n$.

Suppose that A is irreducible. Then B is also irreducible.

Suppose that $\rho(A) = \psi_n$. Then $\rho(B) = \min_{1\leqslant i\leqslant n} r_i(B) = \psi_n$. By Lemma 2.1, $r_1(B) = \cdots = r_n(B) = \psi_n$, and thus from the arguments above, (a) and (b) for $1 \leqslant i \leqslant n-1$ and (c) hold. If $m_1 > m_n$, then for $1 \leqslant i \leqslant t-1$ where t is the smallest integer with $m_t = m_n$, we have $m_i > m_n$, implying that $x_i > 1$, and thus (i)–(iii) follow from (a), (b) for $1 \leqslant i \leqslant n-1$ and (c).

Conversely, if $m_1 = \cdots = m_n$, then by Lemma 2.2, $\rho(A) = m_n = \psi_n$. If (i)–(iii) hold for some t with $2 \leqslant t \leqslant n$, then (a), (b) for $1 \leqslant i \leqslant n-1$ and (c) hold, implying that $r_i(B) = \psi_n$ for $1 \leqslant i \leqslant n$, and thus by Lemma 2.1, $\rho(A) = \rho(B) = \psi_n$. □

Under the notations and conditions of Theorem 2.3, if A is symmetric, then conditions (i)–(iii) hold if and only if conditions (i′) and (ii′) hold:

(i′) $a_{11} = S$ and the off-diagonal elements of A in the first row and column are equal to T,

(ii′) $r_2(A) = \cdots = r_n(A)$.

In [8], the following lower bound for the spectral radius was given.

Theorem 2.4. *(See [8].) Let A be an $n \times n$ nonnegative matrix with row sums $r_1 \geqslant \cdots \geqslant r_n$. Let S be the smallest diagonal element, and T the smallest off-diagonal element of A. Let*

$$\Psi_n = \frac{r_n + S - T + \sqrt{(r_n - S + T)^2 + 4T \sum_{i=1}^{n-1}(r_i - r_n)}}{2}.$$

Then $\rho(A) \geqslant \Psi_n$.

For A_1 as earlier, in notations of Theorem 2.3, $m_1 = 5$, $m_2 = m_3 = m_4 = \frac{23}{5}$, $S = 0$, $T = 1$ and $c = \frac{3}{5}$, implying that $\psi_4 = 4.6458$, and thus $\rho(A_1) \geqslant 4.6458$. For A_1', in notations of Theorem 2.4, $r_1 = r_2 = r_3 = 5$, $r_4 = 3$, $S = 0$ and $T = 1$, implying that $\Psi_4 = 4.1623$, and thus $\rho(A_1) \geqslant 4.1623$. Thus the lower bound in Theorem 2.3 is greater than that in Theorem 2.4 for A_1.

Now consider

$$A_3 = \begin{pmatrix} 0 & 1.9 & 1.9 & 1.9 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 4 & 2 & 2 & 0 \end{pmatrix}.$$

In notations of Theorem 2.3, we have $m_1 = \frac{20}{3}$, $m_2 = m_3 = \frac{197}{30}$, $m_4 = 5.85$, $S = 0$, $T = 1.9$ and $c = 0.7125$, implying that $\psi_4 = 6.2506$, and thus $\rho(A_3) \geqslant 6.2506$. Obviously, A_3 is permutation similar to

$$A_3' = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 1.9 & 1.9 & 1.9 & 0 \end{pmatrix},$$

and thus $\rho(A_3) = \rho(A_3')$. In notations of Theorem 2.4, for A_3', we have $r_1 = 8$, $r_2 = r_3 = 6$, $r_4 = 5.7$, $S = 0$ and $T = 1.9$, implying that $\Psi_4 = 6.3665$, and thus $\rho(A_3) \geqslant 6.3665$. Thus the lower bound in Theorem 2.3 is smaller than that in Theorem 2.4 for A_3.

3. Spectral radius of adjacency and signless Laplacian matrices

Let G be an n-vertex graph without isolated vertices, where $V(G) = \{v_1, \ldots, v_n\}$. For $1 \leqslant i \leqslant n$, recall that $m_i(A(G)) = \frac{\sum_{v_i v_j \in E(G)} d_j}{d_i}$ is the average 2-degree of vertex v_i in G, and note that $m_i(Q(G)) = d_i + \frac{\sum_{v_i v_j \in E(G)} d_j}{d_i}$, which we call the signless Laplacian average 2-degree of vertex v_i in G. Let Δ and δ be respectively the maximal and minimal degrees of G.

The following result has been given by Huang and Weng [10] for a connected graph.

Theorem 3.1. *Let G be a graph on n vertices without isolated vertices with average 2-degrees $m_1 \geqslant \cdots \geqslant m_n$. Then for $1 \leqslant l \leqslant n$,*

$$\rho\big(A(G)\big) \leqslant \frac{m_l - \frac{\Delta}{\delta} + \sqrt{(m_l + \frac{\Delta}{\delta})^2 + 4\frac{\Delta}{\delta}\sum_{i=1}^{l-1}(m_i - m_l)}}{2}.$$

Moreover, if G is connected, then equality holds if and only if $m_1 = \cdots = m_n$.

Proof. We apply Theorem 2.1 to $A(G)$. Since $M = 0$, $N = 1$ and $b = \frac{\Delta}{\delta}$, we have the desired upper bound for $\rho(A(G))$. Suppose that G is connected. Then $A(G)$ is irreducible and symmetric. Thus the upper bound is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $A(G)$ satisfies the following conditions (a) and (b):

(a) the off-diagonal elements of $A(G)$ in the first row and column are equal to 1,
(b) $r_2(A(G)) = \cdots = r_n(A(G))$.

From (a), we have $d_1 = n - 1$, and from (b) and by noting that $m_1 > m_n$, we have $d_2 = \cdots = d_n < n - 1$. If (a) and (b) hold, then for $2 \leqslant i \leqslant n$, $m_i = \frac{(n-1)+(d_i-1)d_i}{d_i} = d_i + \frac{n-1-d_i}{d_i} > d_i = m_1$, a contradiction. □

From Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let G be a graph on n vertices without isolated vertices with average 2-degrees $m_1 \geqslant \cdots \geqslant m_n$. Then for $1 \leqslant l \leqslant n$,*

$$\rho\big(A(G)\big) \leqslant \frac{m_l - \frac{\Delta}{\delta} + \sqrt{(m_l + \frac{\Delta}{\delta})^2 + 4\frac{\Delta}{\delta}(l-1)(m_1 - m_l)}}{2}.$$

Moreover, if G is connected, then equality holds if and only if $m_1 = \cdots = m_n$.

Setting $l = 2$ in the previous corollary, we have the following result, which has been given by Chen et al. [4] for a connected graph.

Corollary 3.2. *Let G be a graph on n vertices without isolated vertices with average 2-degrees $m_1 \geqslant \cdots \geqslant m_n$. Then*

$$\rho\big(A(G)\big) \leqslant \frac{m_2 - \frac{\Delta}{\delta} + \sqrt{(m_2 - \frac{\Delta}{\delta})^2 + 4m_1\frac{\Delta}{\delta}}}{2}.$$

Moreover, if G is connected, then equality holds if and only if $m_1 = \cdots = m_n$.

Theorem 3.2. *Let G be a graph on n vertices without isolated vertices with signless Laplacian average 2-degrees $m_1 \geqslant \cdots \geqslant m_n$. Then for $1 \leqslant l \leqslant n$,*

$$\rho\big(Q(G)\big) \leqslant \frac{m_l + \Delta - \frac{\Delta}{\delta} + \sqrt{(m_l - \Delta + \frac{\Delta}{\delta})^2 + 4\frac{\Delta}{\delta}\sum_{i=1}^{l-1}(m_i - m_l)}}{2}.$$

Moreover, if G is connected, then equality holds if and only if $m_1 = \cdots = m_n$ or $d_1 = n-1 > d_2 = \cdots = d_n$.

Proof. We apply Theorem 2.1 to $Q(G)$. Since $M = \Delta$, $N = 1$ and $b = \frac{\Delta}{\delta}$, we have the desired upper bound for $\rho(Q(G))$. Suppose that G is connected. Then $Q(G)$ is irreducible and symmetric. Thus the upper bound is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $Q(G) = (q_{ij})$ satisfies the following conditions (a) and (b):

(a) $q_{11} = \Delta$ and the off-diagonal elements of $Q(G)$ in the first row and column are equal to 1,
(b) $r_2(Q(G)) = \cdots = r_n(Q(G))$,

or equivalently, either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $d_1 = n-1 > d_2 = \cdots = d_n$. □

We give an example showing that the second condition of the equality case in Theorem 3.2 may occur. Let G be the graph obtained by adding two nonadjacent edges to the 5-vertex star. Then

$$Q(G) = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

By direct computation, $\rho(Q(G)) = 5.5616$. In notations of Theorem 3.2, $m_1 = 6$, $m_2 = m_3 = m_4 = m_5 = 5$, $\Delta = 4$ and $\delta = 2$. Let ϕ_l be the upper bound for $\rho(Q(G))$ in Theorem 3.2, where $1 \leqslant l \leqslant 5$. Then $\phi_1 = 6$ and $\phi_2 = \phi_3 = \phi_4 = \phi_5 = 5.5616$, implying that $\rho(Q(G)) = \phi_2 = \cdots = \phi_5$.

Let G be a graph such that its line graph L_G has no isolated vertices. By upper bounds for $\rho(A(L_G))$ using the average 2-degrees of L_G and the fact that $\rho(Q(G)) = 2 + \rho(A(L_G))$, we may also have upper bounds for $\rho(Q(G))$ using the average 2-degrees of L_G (cf. [4]).

4. Spectral radius of distance-based matrices

Let G be an n-vertex connected graph, where $V(G) = \{v_1, \dots, v_n\}$.

For $1 \leqslant i \leqslant n$, $m_i(D(G)) = \frac{\sum_{j=1}^{n} d_{ij} D_j}{D_i}$, which we call the average 2-transmission of vertex v_i in G, and $m_i(\mathcal{Q}(G)) = D_i + \frac{\sum_{j=1}^{n} d_{ij} D_j}{D_i}$, which we call the signless Laplacian average 2-transmission of vertex v_i in G.

Let $\mathcal{D}$ be the diameter, which is the maximal distance between any two vertices, of G. Let Ω and ω be respectively the maximal and minimal transmissions of G.

Theorem 4.1. *Let G be a connected graph on $n \geqslant 2$ vertices with average 2-transmissions $m_1 \geqslant \cdots \geqslant m_n$. For $1 \leqslant l \leqslant n$,*

$$\rho\big(D(G)\big) \leqslant \frac{m_l - \mathcal{D}\frac{\Omega}{\omega} + \sqrt{(m_l + \mathcal{D}\frac{\Omega}{\omega})^2 + 4\mathcal{D}\frac{\Omega}{\omega}\sum_{i=1}^{l-1}(m_i - m_l)}}{2}$$

with equality if and only if $m_1 = \cdots = m_n$.

Proof. We apply Theorem 2.1 to $D(G)$. Since $M = 0$, $N = \mathcal{D}$ and $b = \frac{\Omega}{\omega}$, the desired upper bound for $\rho(D(G))$ follows, and it is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $D(G)$ satisfies the following conditions (a) and (b):

(a) the off-diagonal elements of $D(G)$ in the first row and column are equal to $\mathcal{D}$,
(b) $r_2(D(G)) = \cdots = r_n(D(G))$.

Since there is at least an element 1 in each row of $D(G)$, (a) implies that $\mathcal{D} = 1$, and thus G is the n-vertex complete graph, for which $m_1 > m_n$ is impossible. $\square$

Theorem 4.2. *Let G be a connected graph on $n \geqslant 2$ vertices with average 2-transmissions $m_1 \geqslant \cdots \geqslant m_n$. Then*

$$\rho\big(D(G)\big) \geqslant \frac{m_n - \frac{\omega}{\Omega} + \sqrt{(m_n + \frac{\omega}{\Omega})^2 + 4\frac{\omega}{\Omega}\sum_{i=1}^{n-1}(m_i - m_n)}}{2}$$

with equality if and only if $m_1 = \cdots = m_n$ or $D_1 = n - 1 < D_2 = \cdots = D_n$.

Proof. We apply Theorem 2.3 to $D(G)$. Since $S = 0$, $T = 1$ and $c = \frac{\omega}{\Omega}$, we have the desired lower bound for $\rho(D(G))$, which is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $D(G)$ satisfies the following conditions (a) and (b):

(a) the off-diagonal elements of $D(G)$ in the first row and column are equal to 1,
(b) $r_2(D(G)) = \cdots = r_n(D(G))$,

or equivalently, either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $D_1 = n - 1 < D_2 = \cdots = D_n$. $\square$

Let G be the 4-vertex star. Obviously, $D(G) = A_1$ (as earlier in Section 2). By a direct calculation, $\rho(D(G))$ is equal to the lower bound given in Theorem 4.2. This shows that the second condition of the equality case in Theorem 4.2 may occur.

By similar arguments as for the distance matrix above, we have

Theorem 4.3. *Let G be a connected graph on $n \geqslant 2$ vertices with signless Laplacian average 2-transmissions $m_1 \geqslant \cdots \geqslant m_n$. Then for $1 \leqslant l \leqslant n$,*

$$\rho\big(\mathcal{Q}(G)\big) \leqslant \frac{m_l + \Omega - \mathcal{D}\frac{\Omega}{\omega} + \sqrt{(m_l - \Omega + \mathcal{D}\frac{\Omega}{\omega})^2 + 4\mathcal{D}\frac{\Omega}{\omega}\sum_{i=1}^{l-1}(m_i - m_l)}}{2}$$

with equality if and only if $m_1 = \cdots = m_n$.

Theorem 4.4. *Let G be a connected graph on $n \geqslant 2$ vertices with signless Laplacian average 2-transmissions $m_1 \geqslant \cdots \geqslant m_n$. Then*

$$\rho\big(\mathcal{Q}(G)\big) \geqslant \frac{m_n + \omega - \frac{\omega}{\Omega} + \sqrt{(m_n - \omega + \frac{\omega}{\Omega})^2 + 4\frac{\omega}{\Omega}\sum_{i=1}^{n-1}(m_i - m_n)}}{2}$$

with equality if and only if $m_1 = \cdots = m_n$.

Proof. We apply Theorem 2.3 to $\mathcal{Q}(G)$. Since $S = \omega$, $T = 1$ and $c = \frac{\omega}{\Omega}$, we have the desired lower bound for $\rho(\mathcal{Q}(G))$, which is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $\mathcal{Q}(G) = (\theta_{ij})$ satisfies the following conditions (a) and (b):

(a) $\theta_{11} = \omega$ and the off-diagonal elements of $\mathcal{Q}(G)$ in the first row and column are equal to 1,
(b) $r_2(\mathcal{Q}(G)) = \cdots = r_n(\mathcal{Q}(G))$.

From (a), we have $D_1 = n - 1$, and from (b) and by noting that $m_1 > m_n$, we have $D_2 = \cdots = D_n > n - 1$. If (a) and (b) hold, then for $2 \leqslant i \leqslant n$, $m_1 = n - 1 + D_i$ and $m_i = D_i + \frac{(n-1)+D_i(D_i-1)}{D_i} = 2D_i + \frac{n-1-D_i}{D_i}$, and thus $m_1 - m_i = n - 1 - D_i - \frac{n-1-D_i}{D_i} = (n - 1 - D_i)(1 - \frac{1}{D_i}) < 0$, a contradiction. $\square$

For $1 \leqslant i \leqslant n$, $m_i(R(G)) = \frac{\sum_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \frac{1}{d_{ij}} R_j}{R_i}$ where $R_i = r_i(R(G)) = \sum_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \frac{1}{d_{ij}}$, which we call the reciprocal average 2-transmission of vertex v_i in G.

Let $R = \max\{R_i\colon 1 \leqslant i \leqslant n\}$ and $r = \min\{R_i\colon 1 \leqslant i \leqslant n\}$.

Theorem 4.5. *Let G be a connected graph on $n \geqslant 2$ vertices with reciprocal average 2-transmissions $m_1 \geqslant \cdots \geqslant m_n$. For $1 \leqslant l \leqslant n$,*

$$\rho\big(R(G)\big) \leqslant \frac{m_l - \frac{R}{r} + \sqrt{(m_l + \frac{R}{r})^2 + 4\frac{R}{r}\sum_{i=1}^{l-1}(m_i - m_l)}}{2}$$

with equality if and only if $m_1 = \cdots = m_n$.

Proof. We apply Theorem 2.1 to $R(G)$. Since $M = 0$, $N = 1$ and $b = \frac{R}{r}$, we have the desired upper bound for $\rho(R(G))$, which is attained if and only if either $m_1 = \cdots = m_n$ or (if $m_1 > m_n$, then) $R(G)$ satisfies the following conditions (a) and (b):

(a) the off-diagonal elements of $R(G)$ in the first row and column are equal to 1,
(b) $r_2(R(G)) = \cdots = r_n(R(G))$.

From (a), we have $R_1 = n - 1$, and from (b) and by noting that $m_1 > m_n$, we have $R_2 = \cdots = R_n < n - 1$. If (a) and (b) hold, then for $2 \leqslant i \leqslant n$, $m_i = \frac{(n-1)+R_i(R_i-1)}{R_i} = R_i + \frac{n-1-R_i}{R_i} > R_i = m_1$, a contradiction. □

Acknowledgements

This work was supported by the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20124407110002) and the Scientific Research Foundation of Graduate School of South China Normal University (No. 2013kyjj003).

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