Fuzzy Systems

Adopted from Debasis Samanta

Introduction

- Fuzzy logic is a mathematical language to express something.

  This means it has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known

  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate logic** (operations on well formed formulae (wff), also called predicate propositions)

- **Fuzzy logic deals with Fuzzy set.**

Introduction

- First time introduced by Lotfi Abdelli Zadeh (1965), University of California, Berkley, USA (1965).

- He is fondly nick-named as **LAZ**

Introduction

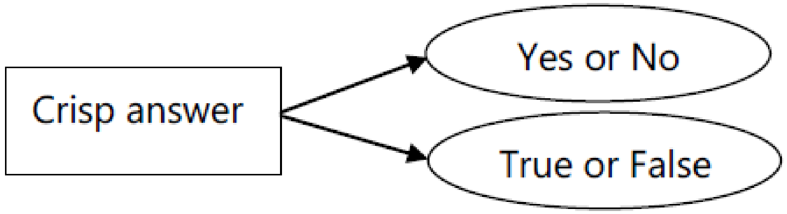
1. Dictionary meaning of **fuzzy** is not clear, noisy etc.

   Example: Is the picture on this slide is fuzzy?

2. Antonym of fuzzy is **crisp**

   Example: Are the chips crisp?

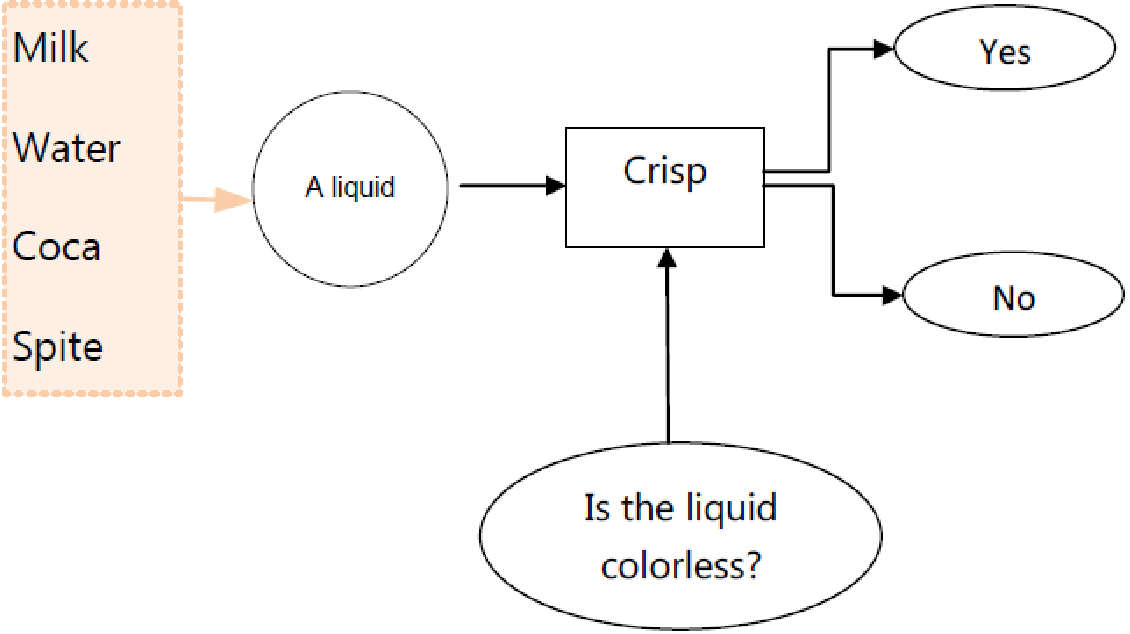
Introduction



Crisp answer

Yes or No

True or False



Milk

Water

Coca

Spite

A liquid

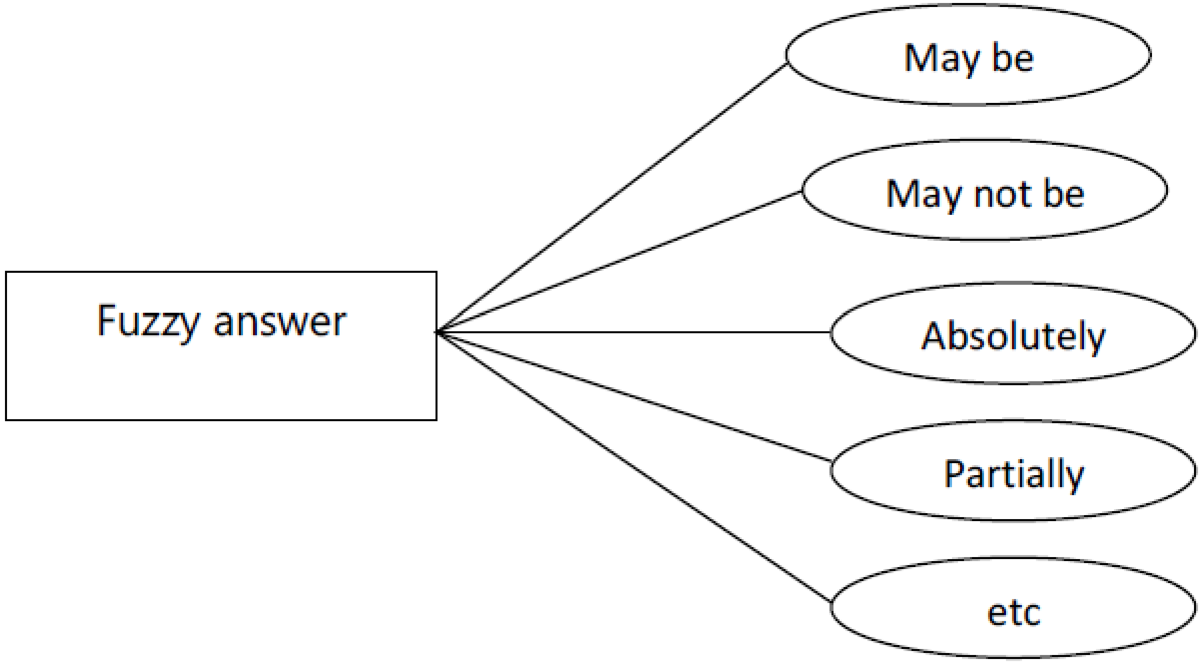
Crisp

Yes

No

Is the liquid colorless?

Introduction


Fuzzy answer

May be

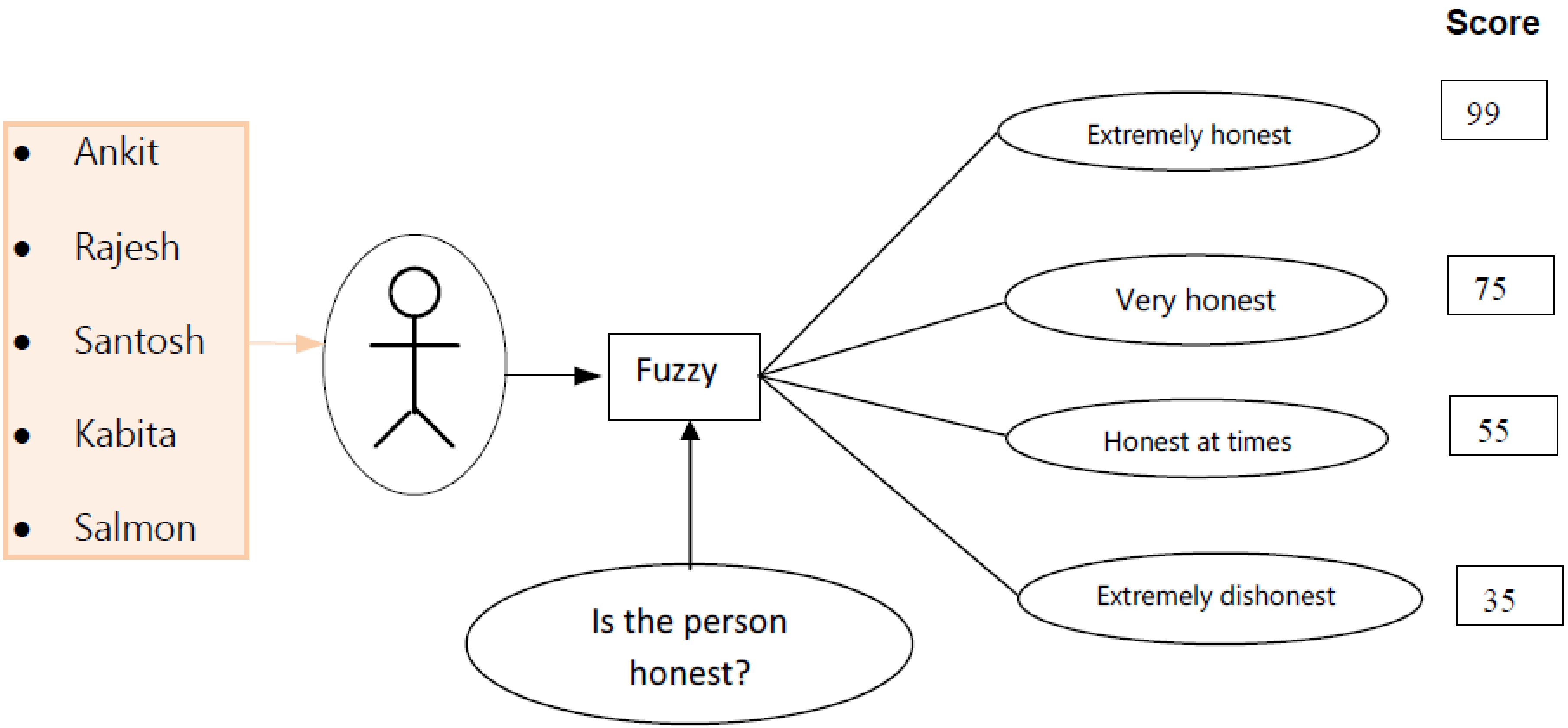
May not be

Absolutely

Partially

etc

Introduction



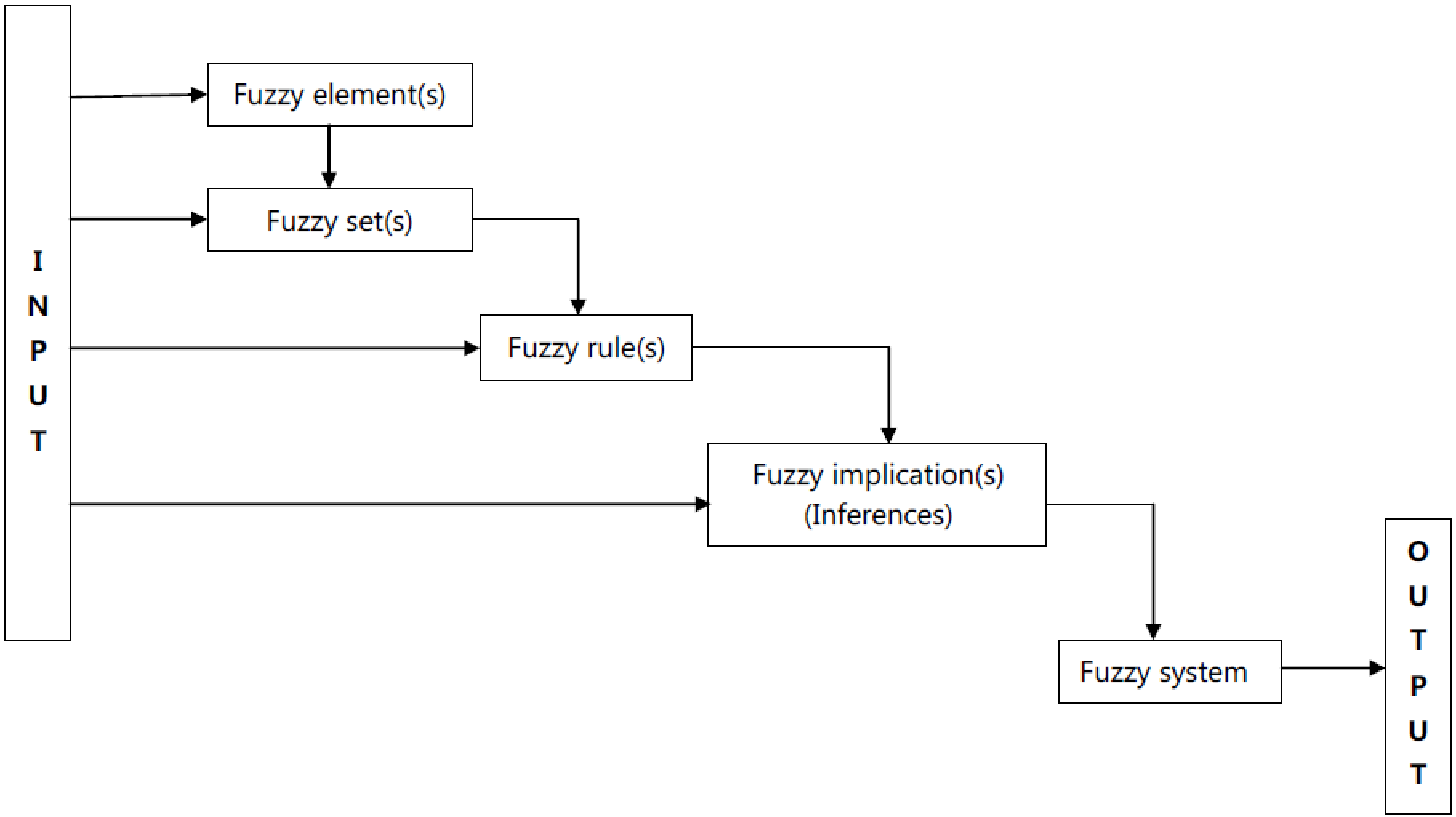
- Ankit
- Rajesh
- Santosh
- Kabita
- Salmon

Fuzzy

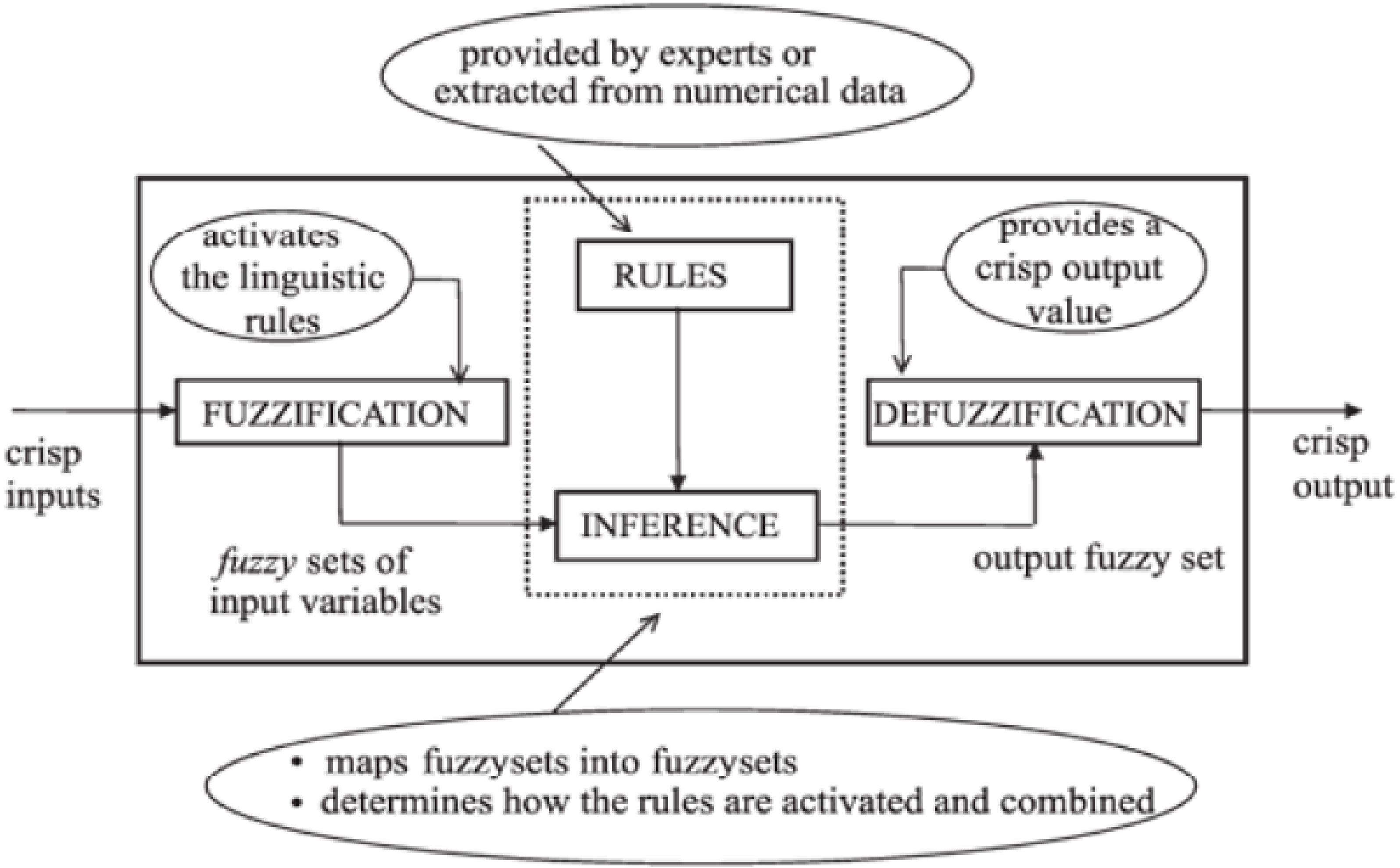
Is the person honest?

	Score
Extremely honest	99
Very honest	75
Honest at times	55
Extremely dishonest	35

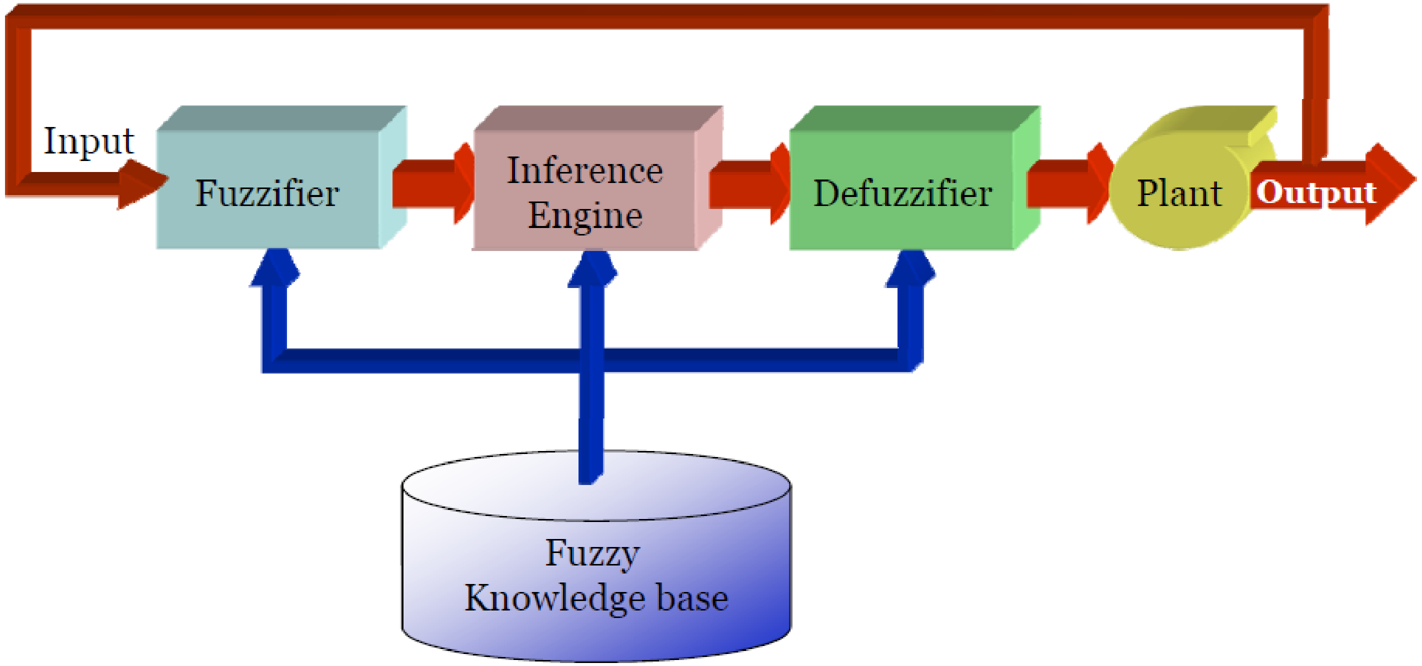
Phases


INPUT
Fuzzy element(s)
Fuzzy set(s)
Fuzzy rule(s)
Fuzzy implication(s)
(Inferences)
Fuzzy system
OUTPUT

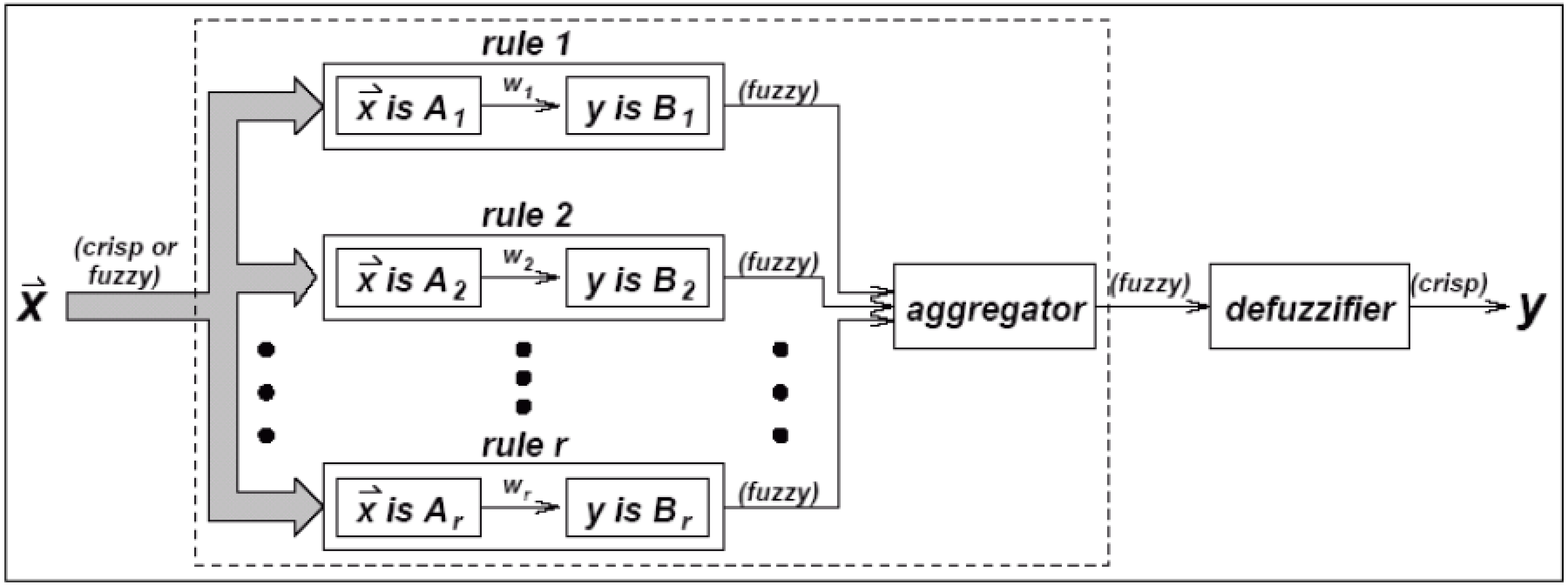
System


provided by experts or extracted from numerical data
activates the linguistic rules
RULES
provides a crisp output value
crisp inputs
FUZZIFICATION
DEFUZZIFICATION
crisp output
INFERENCE
fuzzy sets of input variables
output fuzzy set
• maps fuzzysets into fuzzysets
• determines how the rules are activated and combined

System

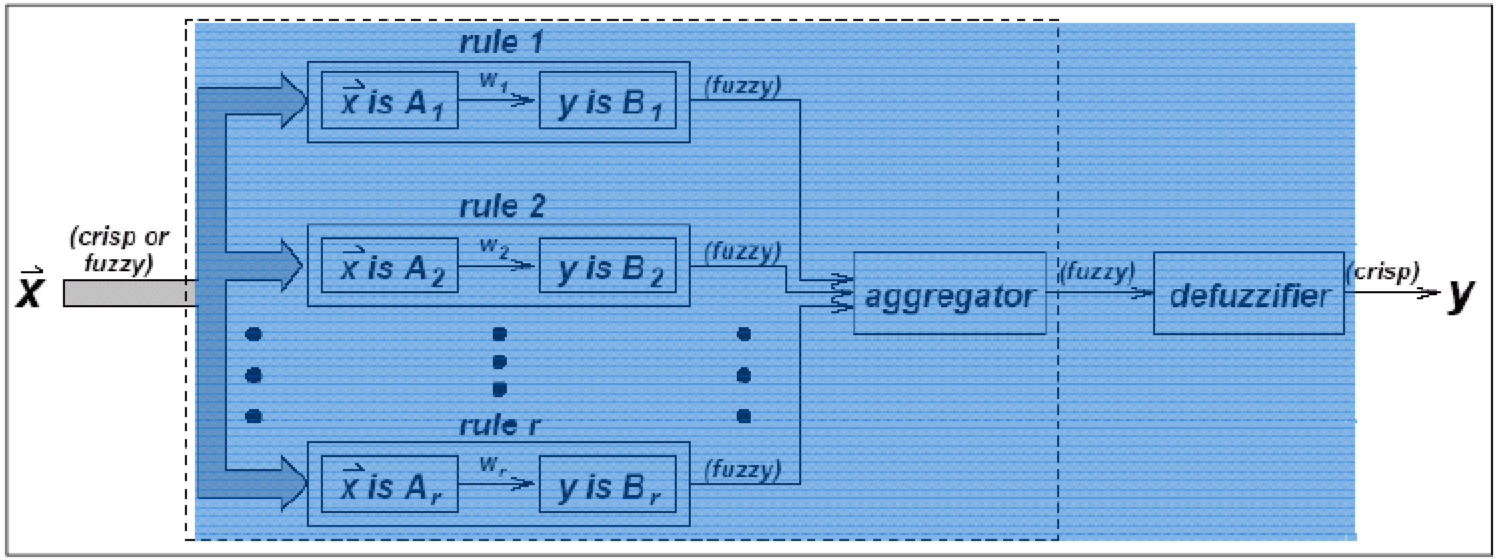

Input
Fuzzifier
Inference Engine
Defuzzifier
Plant
Output
Fuzzy Knowledge base

System


rule 1
$\vec{x}$ is A_1
w_1
y is B_1
(fuzzy)
rule 2
$\vec{x}$ is A_2
w_2
y is B_2
(fuzzy)
rule r
$\vec{x}$ is A_r
w_r
y is B_r
(fuzzy)
$\vec{x}$
(crisp or fuzzy)
aggregator
(fuzzy)
defuzzifier
(crisp)
y

Mapping

In the case of crisp inputs & outputs, a fuzzy inference system implements a nonlinear mapping from its input space to output space.



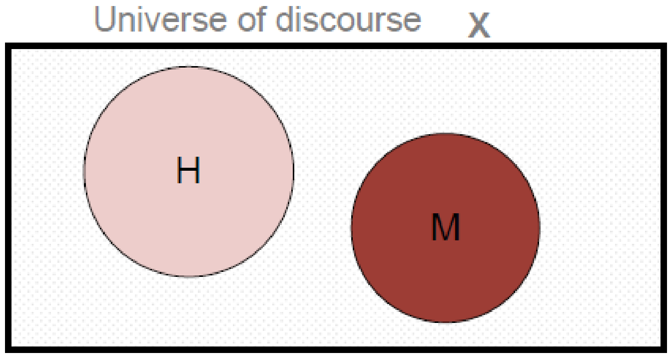
Fuzzy Sets

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{ h_1, h_2, h_3, \ldots , h_L \}$

M = All Muslim population = $\{ m_1, m_2, m_3, \ldots , m_N \}$



Here, All are the sets of finite numbers of individuals.

Such a set is called crisp set.

Fuzzy Sets

Let us discuss about fuzzy set.

X = All students in IT60108.

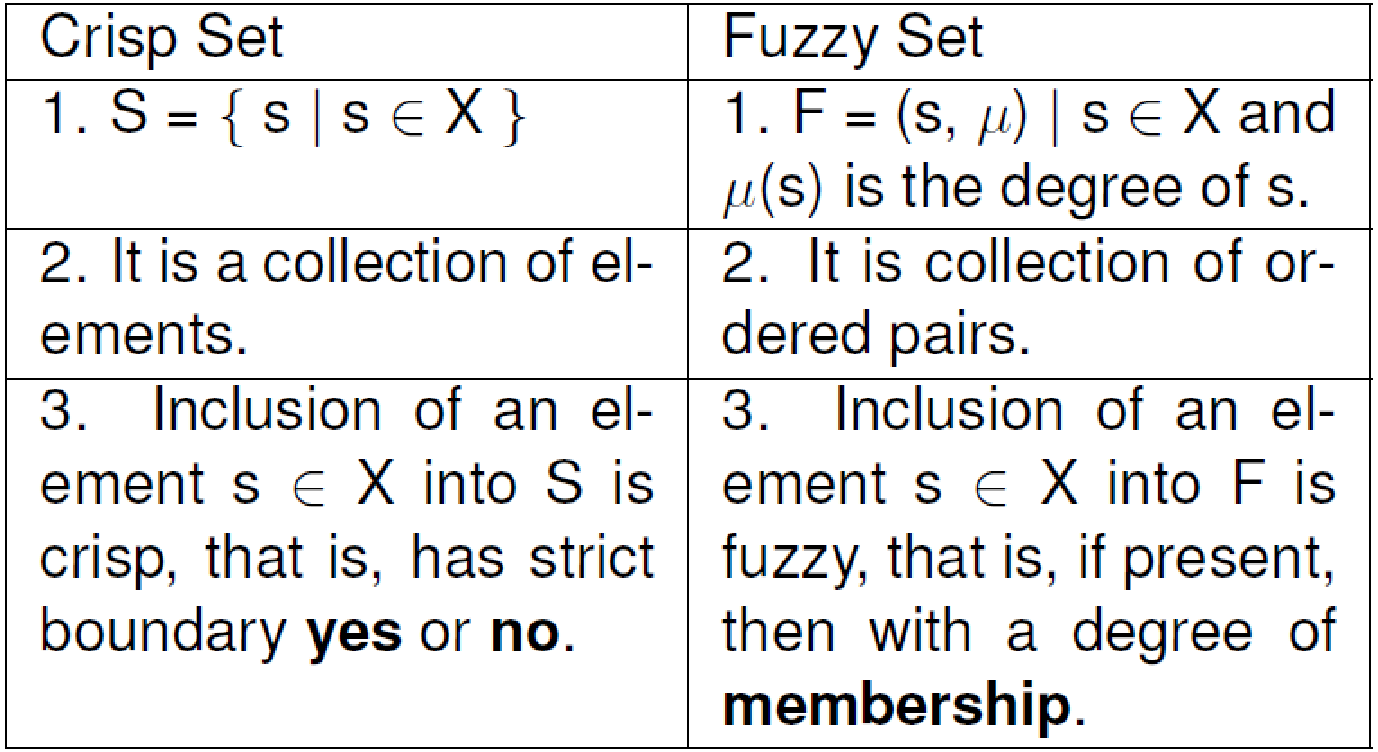
S = All **Good students**.

S = $\{ (s, g) \mid s \in X \}$ and g(s) is a measurement of goodness of the student *s*.

Example:

S = { (Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9) } etc.

Fuzzy Sets



Crisp Set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu) \mid s \in X$ and $\mu(s)$ is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of **membership**.

Fuzzy Sets

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:
H = { $(h_1, 1), (h_2, 1), \ldots , (h_L, 1)$ }

Person = { $(p_1, 1), (p_2, 0), \ldots , (p_N, 1)$ }

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

How to decide the degree of memberships of elements in a fuzzy set?

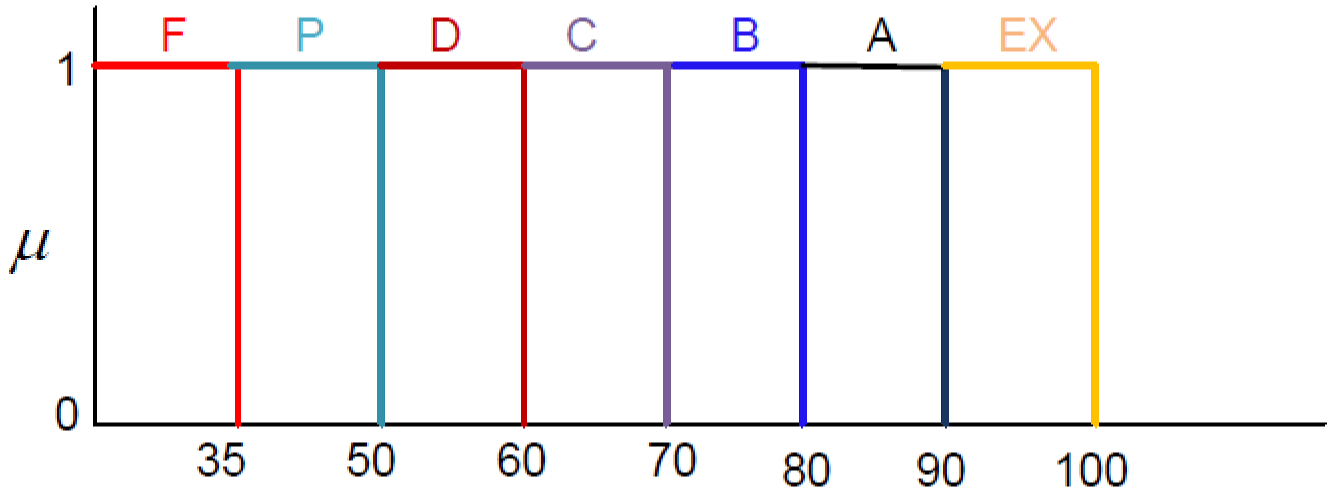
City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
DoM	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

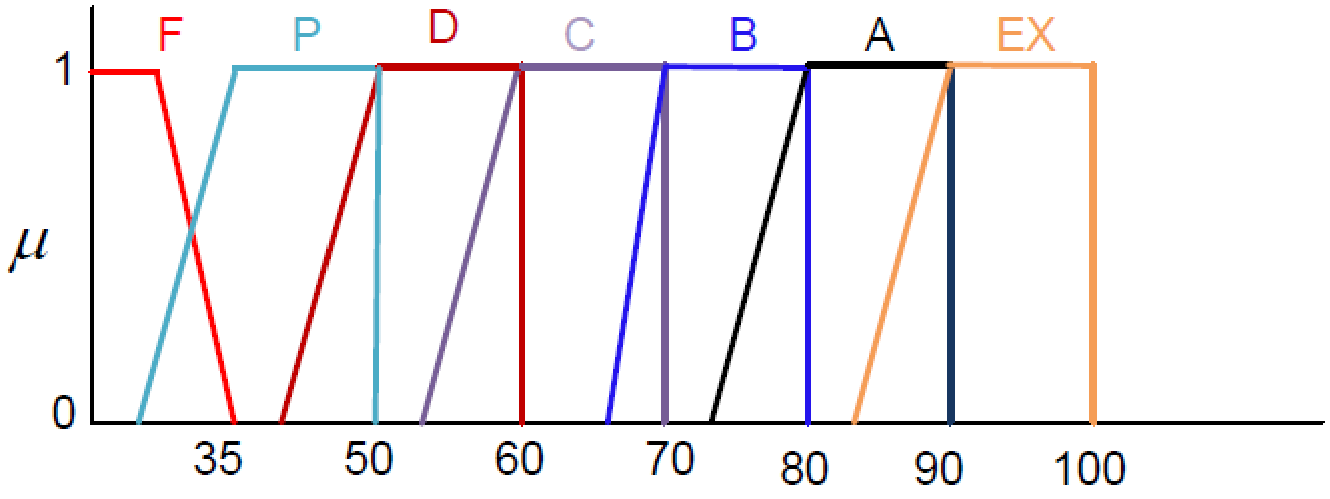
Fuzzy Sets

1. EX = Marks $\geq$ 90
2. A = 80 $\leq$ Marks $<$ 90
3. B = 70 $\leq$ Marks $<$ 80
4. C = 60 $\leq$ Marks $<$ 70
5. D = 50 $\leq$ Marks $<$ 60
6. P = 35 $\leq$ Marks $<$ 50
7. F = Marks $<$ 35

Fuzzy Sets



Fuzzy Sets



Examples

- High Temperature
- Low Pressure

- Color of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Fuzzy Sets

Definition 1: Membership function (and Fuzzy set)

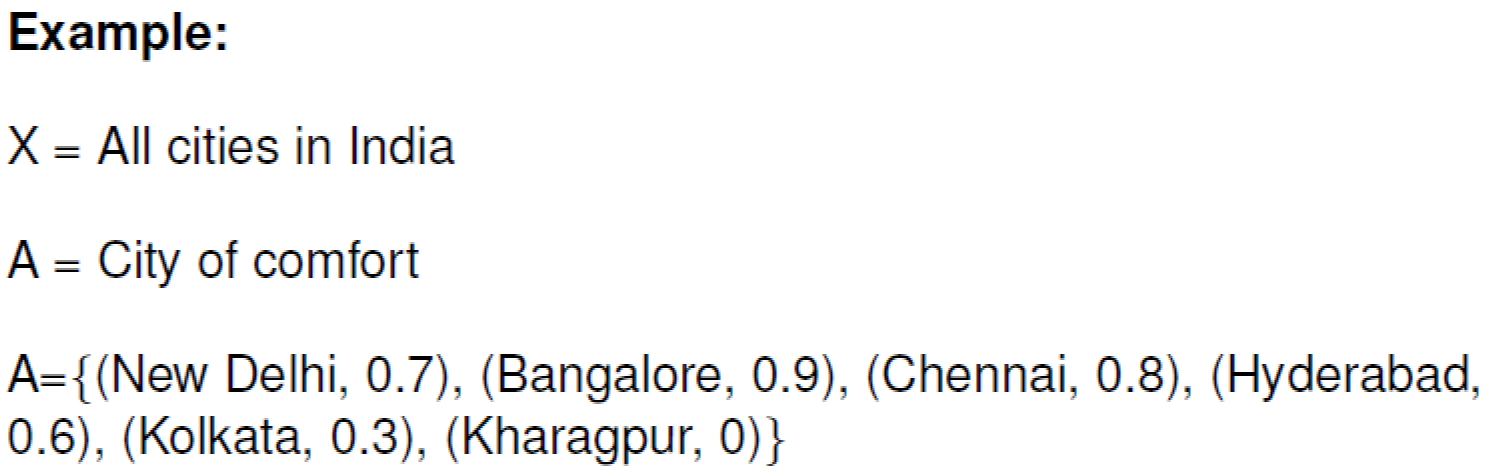
If X is a universe of discourse and x $\in$ X, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where μ_A(x) is called the membership function for the fuzzy set A.

Note:
μ_A(x) map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question:
How (and who) decides μ_A(x) for a Fuzzy set A in X?

Fuzzy Sets



Example:

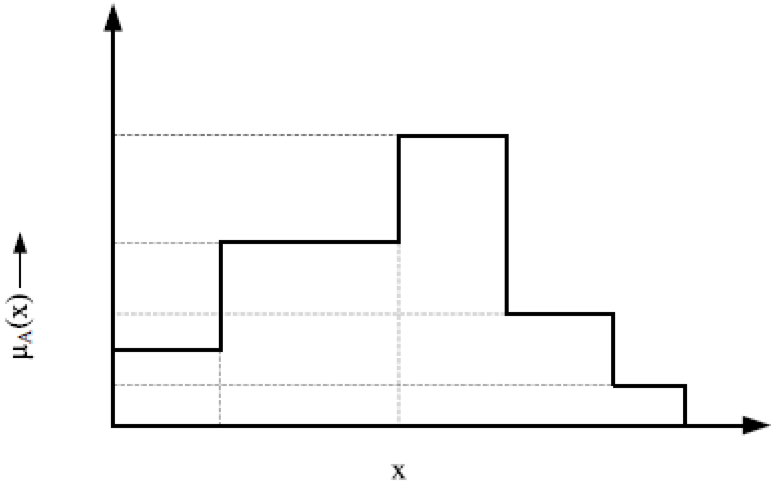
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Fuzzy Sets

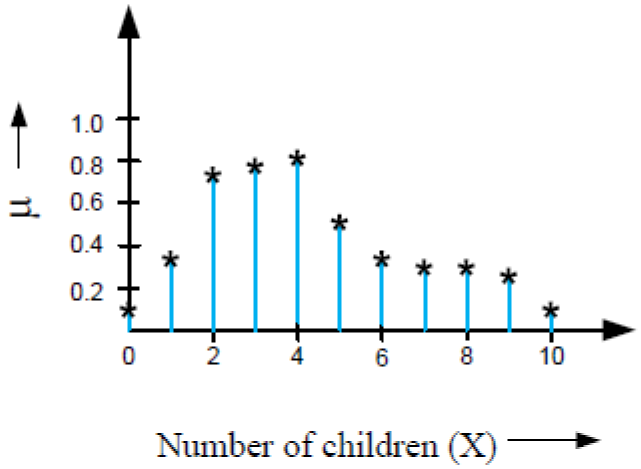
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Fuzzy Sets

Either elements or their membership values (or both) also may be of discrete values.



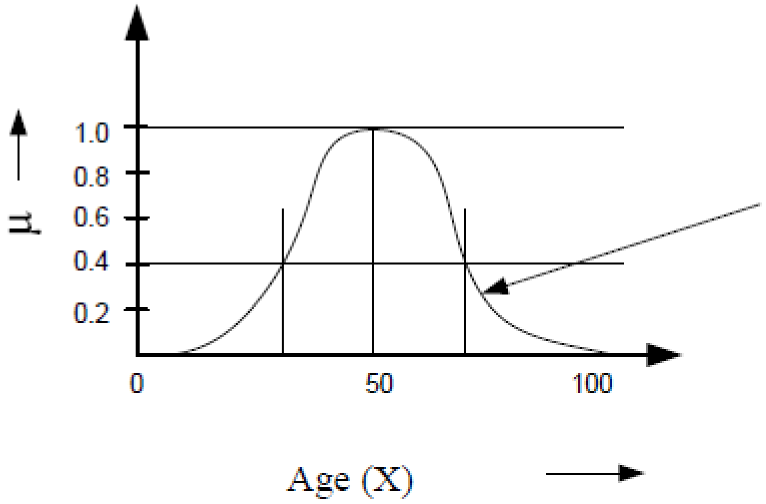
A = "Happy family"

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

Note : X = discrete value

How you measure happiness ??

Fuzzy Sets



$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

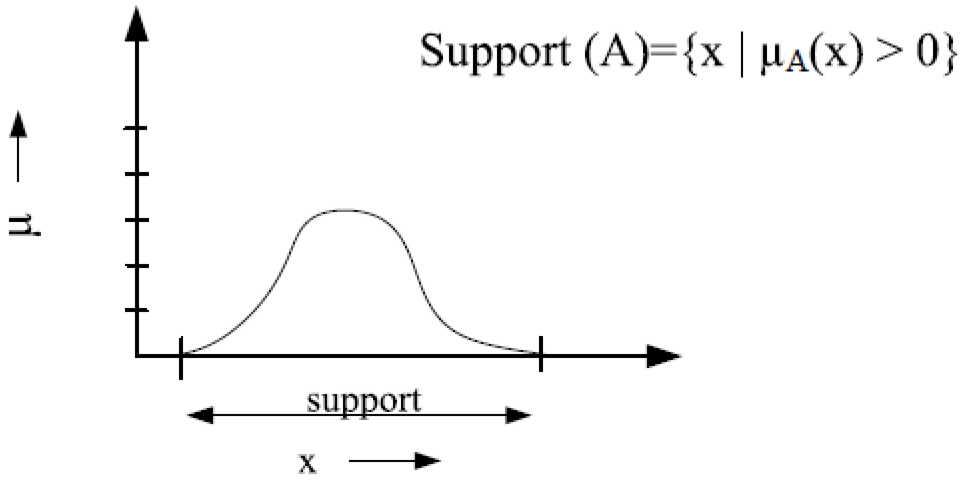
$$B=\{(x,\mu_B(x))\}$$

B = “Middle aged”

Note : x = real value
= R^+

Fuzzy Sets

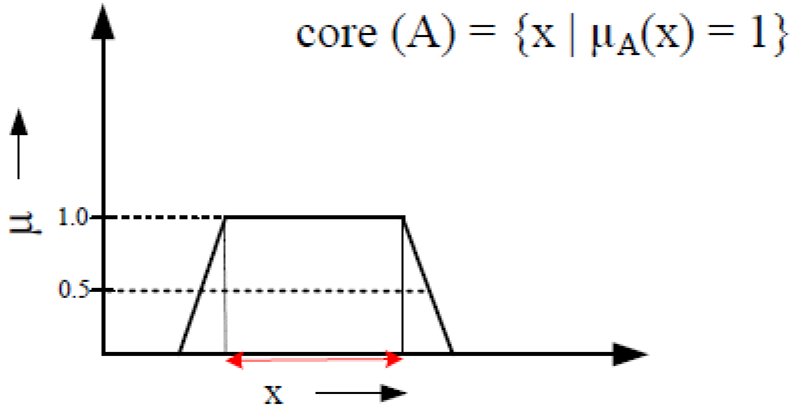
Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



Support (A)=$\{x \mid \mu_A(x) > 0\}$

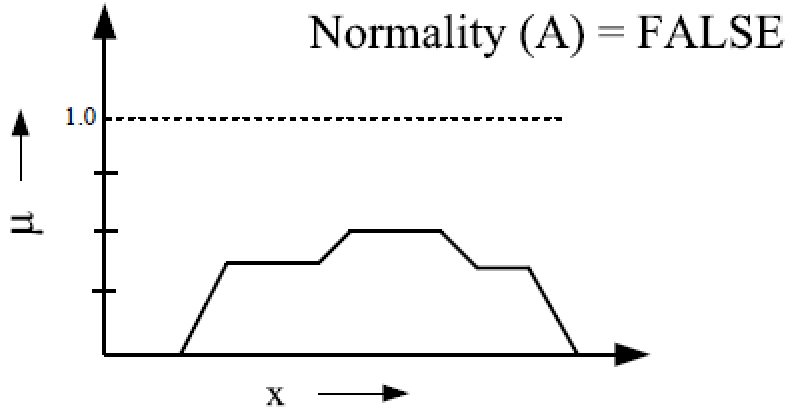
Fuzzy Sets

Core: The core of a fuzzy set *A* is the set of all points *x* in *X* such that $\mu_A(x) = 1$



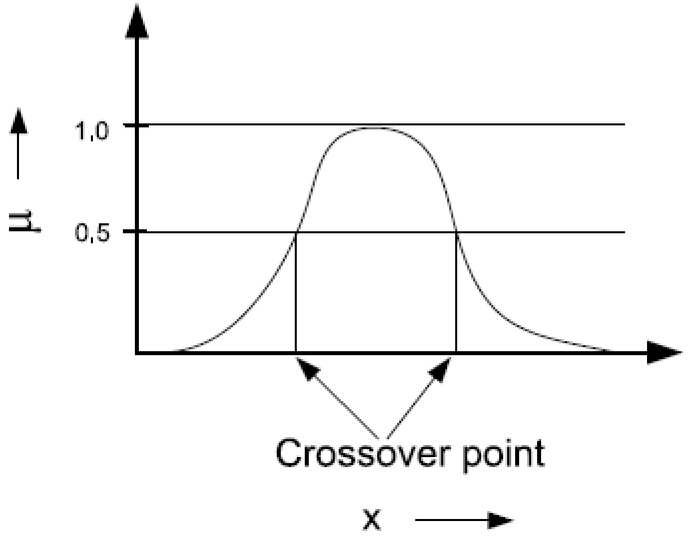
Fuzzy Sets

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.



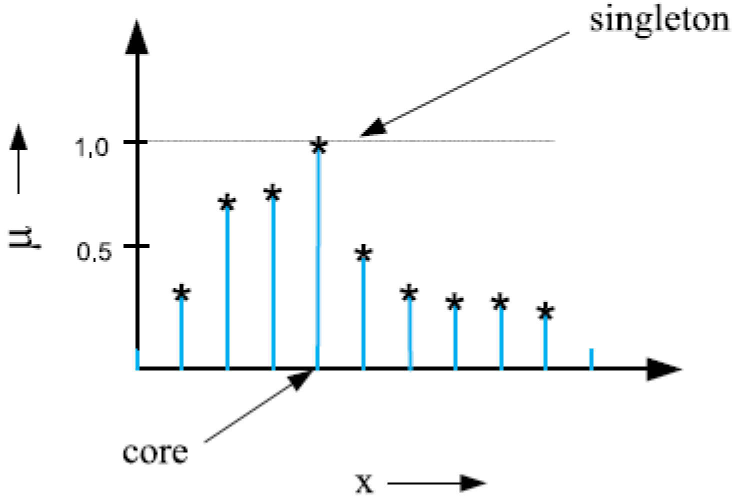
Fuzzy Sets

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is
Crossover $(A) = \{x | \mu_A(x) = 0.5\}$.



Fuzzy Sets

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = |\{ x \mid \mu_A(x) = 1\}| = 1$. Following fuzzy set is not a fuzzy singleton.



Fuzzy Sets

α-cut and strong α-cut :

The α-cut of a fuzzy set *A* is a crisp set defined by

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha \}$$

Strong α-cut is defined similarly :

$$A_\alpha' = \{x \mid \mu_A(x) > \alpha \}$$

Note : Support(A) = A_0' and Core(A) = A_1.

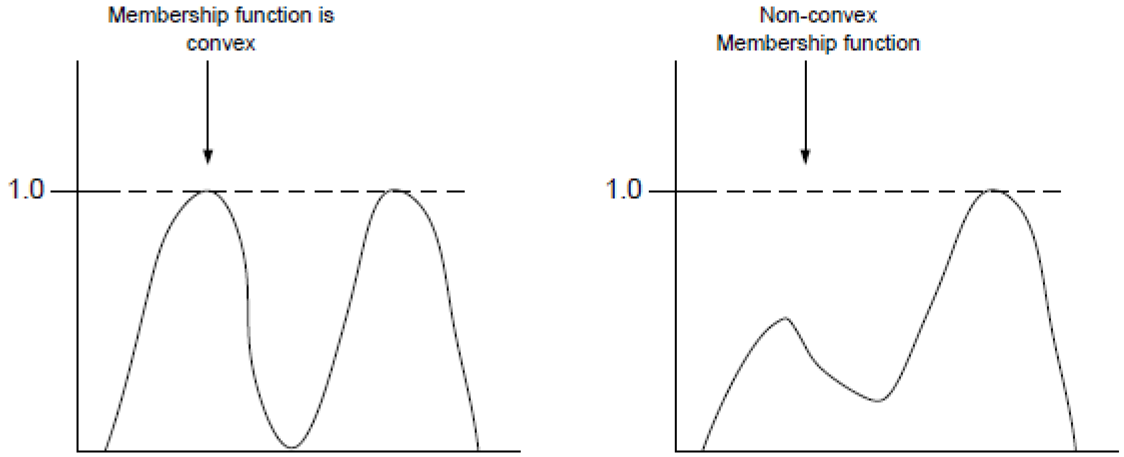
Fuzzy Sets

Convexity : A fuzzy set A is convex if and only if for any x_1 and $x_2 \in X$ and any $\lambda \in [0, 1]$

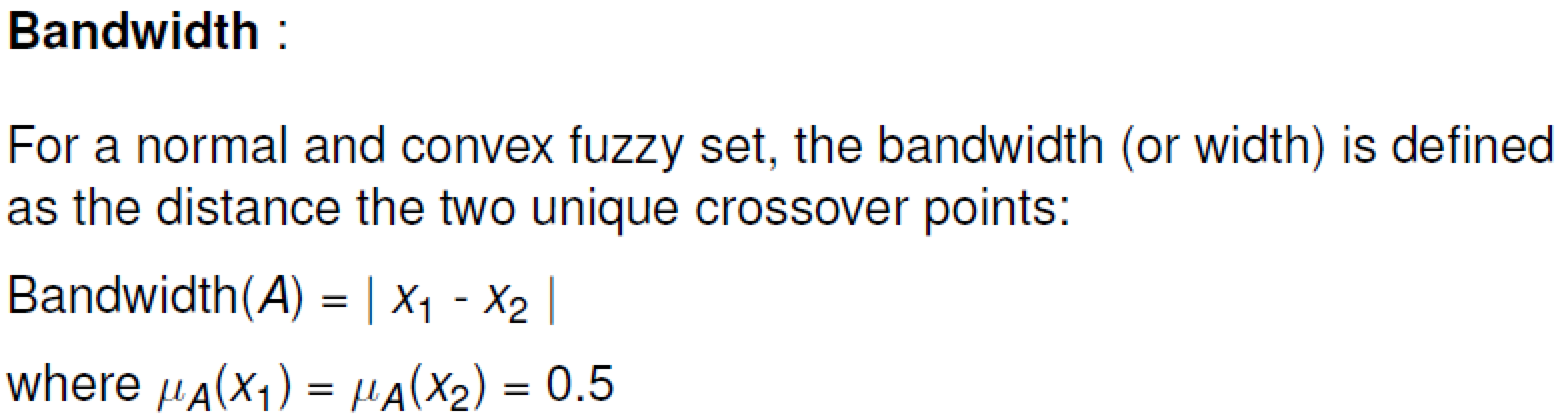
$$\mu_A\,(\lambda x_1 + (1 - \lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Note :

- A is convex if all its α- level sets are convex.
- Convexity $(A_\alpha) \Longrightarrow A_\alpha$ is composed of a single line segment only.



Fuzzy Sets



Bandwidth :

For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

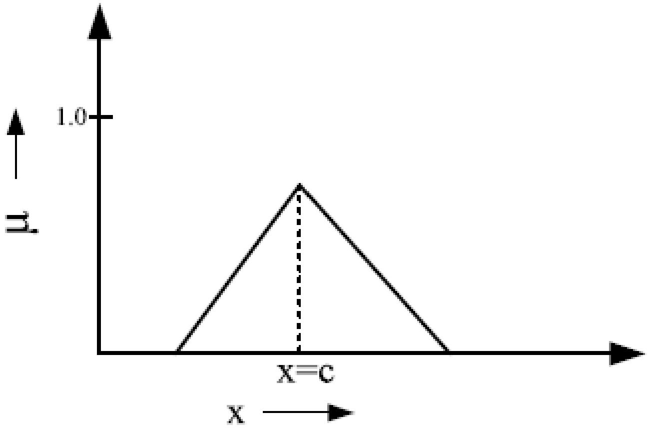
Bandwidth(A) = $| x_1 - x_2 |$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy Sets

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$.



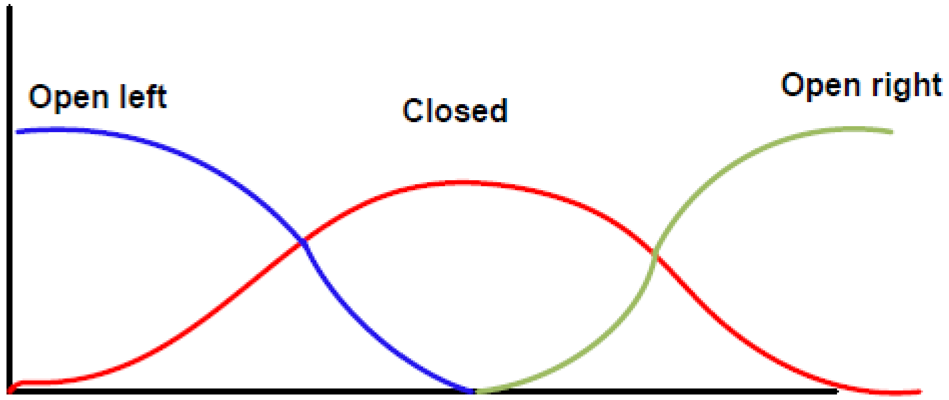
Fuzzy Sets

A fuzzy set *A* is

Open left
If $\lim_{x\to-\infty} \mu_A(x) = 1$ and $\lim_{x\to+\infty} \mu_A(x) = 0$

Open right:
If $\lim_{x\to-\infty}\mu_A(x) = 0$ and $\lim_{x\to+\infty} \mu_A(x) = 1$

Closed
If : $\lim_{x\to-\infty} \mu_A(x) = \lim_{x\to+\infty} \mu_A(x) = 0$



Fuzzy vs Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.
Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

Prediction is based on the best guess from experiences.
Forecasting is based on data you have actually recorded and packed from previous job.

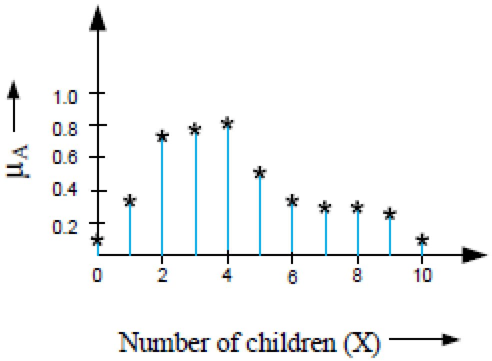
Membership Functions

A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

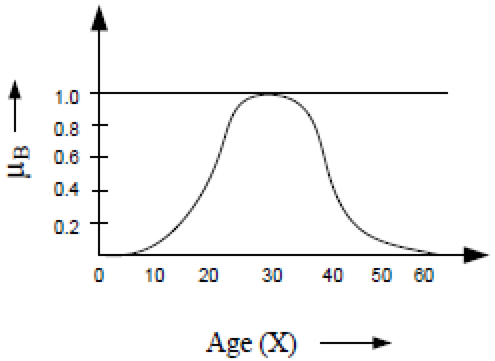
Note: A membership function can be on

(a) a discrete universe of discourse and
(b) a continuous universe of discourse.

Example:



A = Fuzzy set of "Happy family"

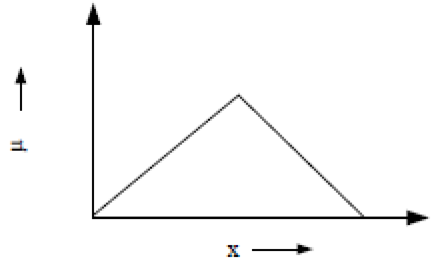


B = "Young age"

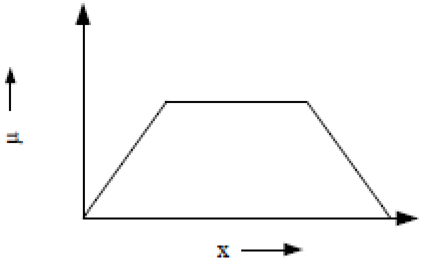
Membership Functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

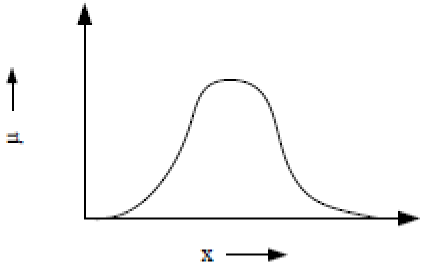
Following figures shows a typical examples of membership functions.



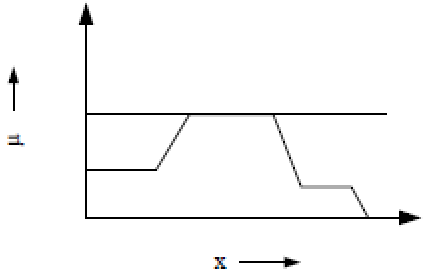
< triangular >



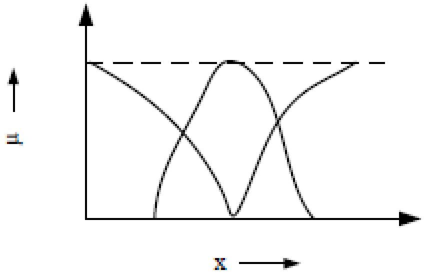
< trapezoidal >



< curve >



< non-uniform >



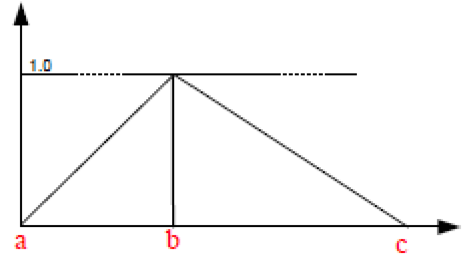
< non-uniform >

Membership Functions

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.

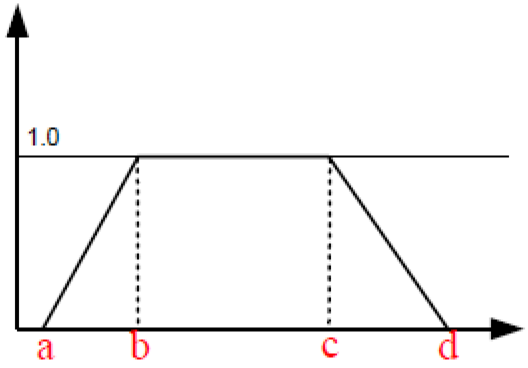
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$



Membership Functions

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

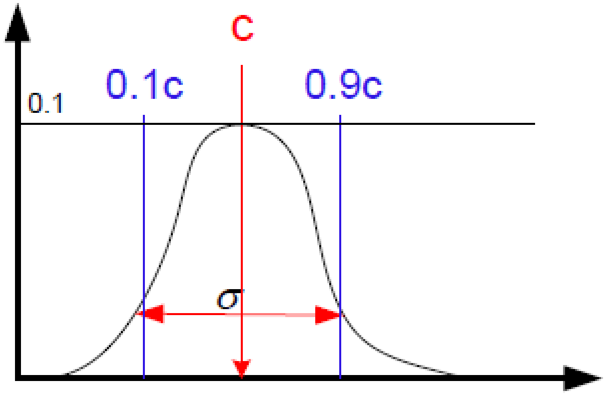
$$trapezoid(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$



Membership Functions

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

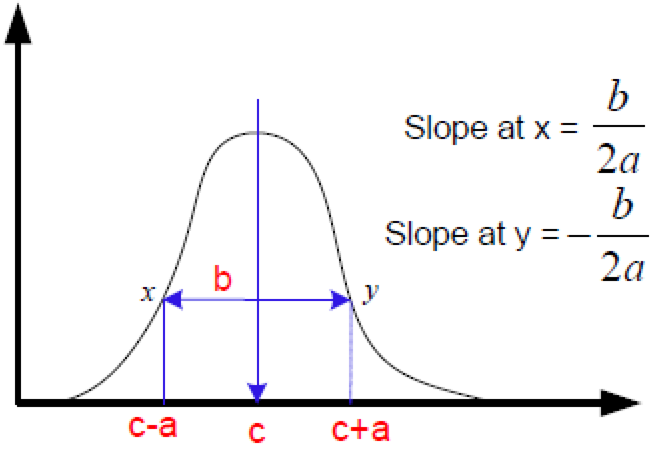
$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2} .$$



Membership Functions

It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

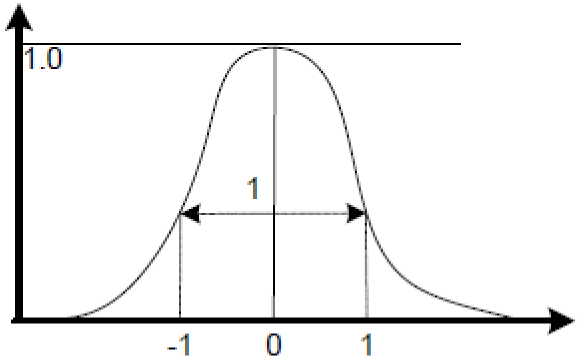


Slope at x = $\frac{b}{2a}$

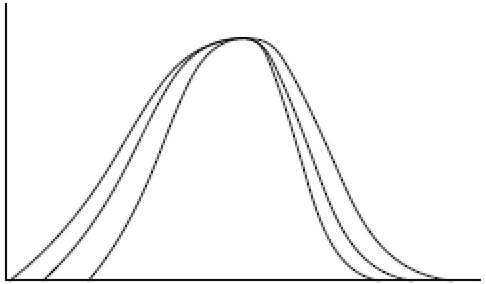
Slope at y = $-\frac{b}{2a}$

Membership Functions

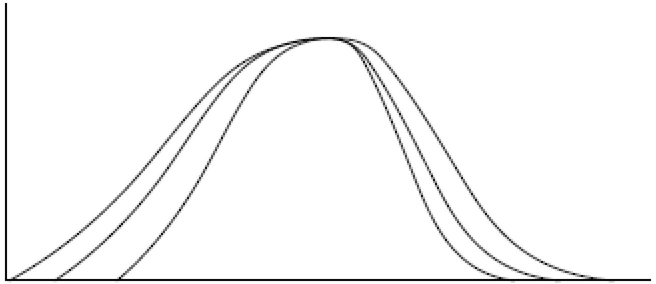
Example: $\mu(x) = \frac{1}{1+x^2}$;
$a = b = 1$ and $c = 0$;



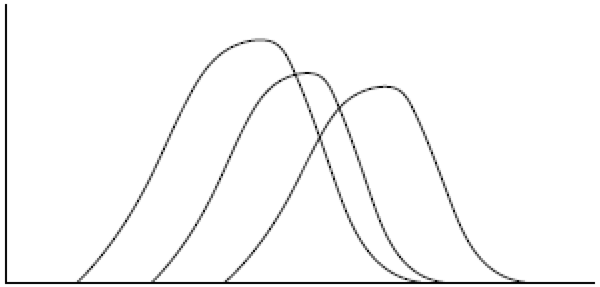
Membership Functions



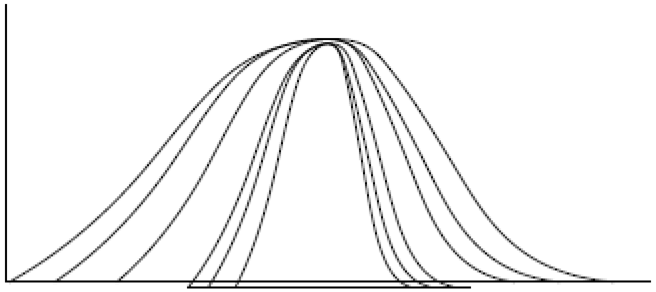
Changing a



Changing b



Changing a

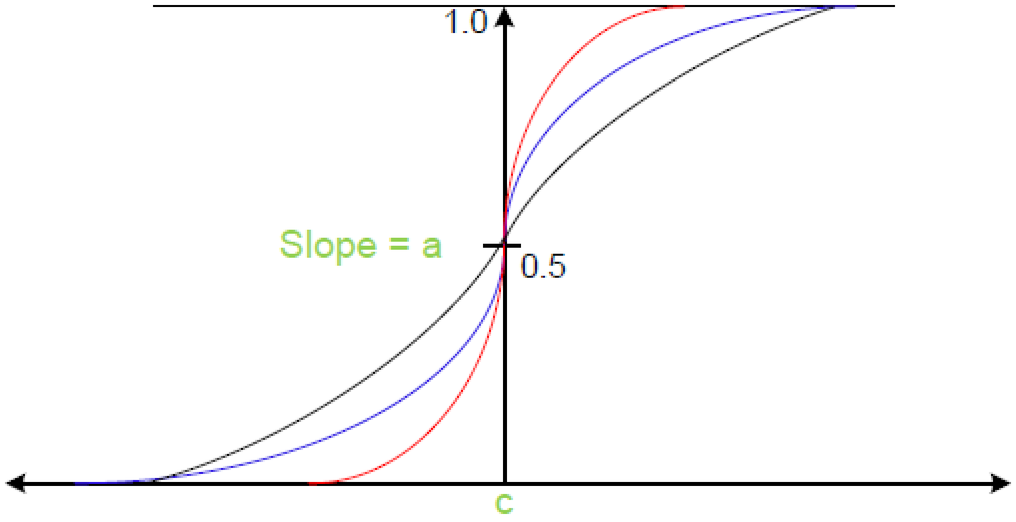


Changing a and b

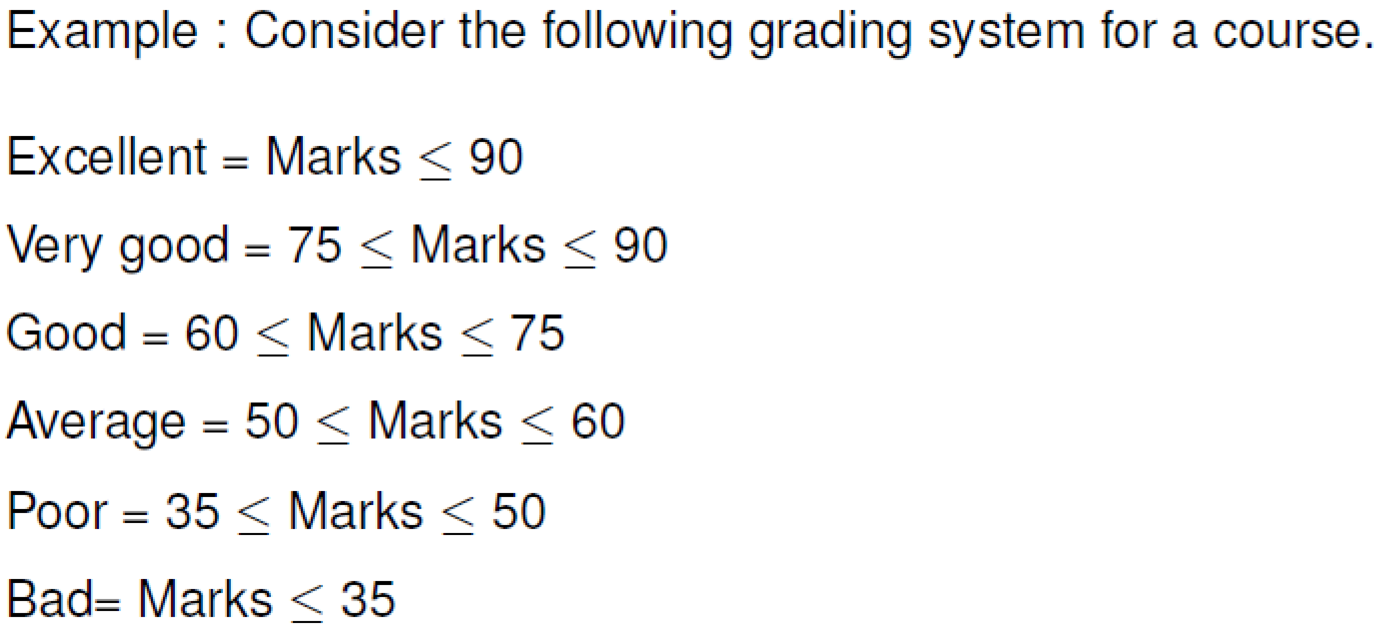
Membership Functions

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;

$$\text{Sigmoid}(x;a,c)=\frac{1}{1+e^{-[\frac{a}{x-c}]}}$$



Membership Functions



Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

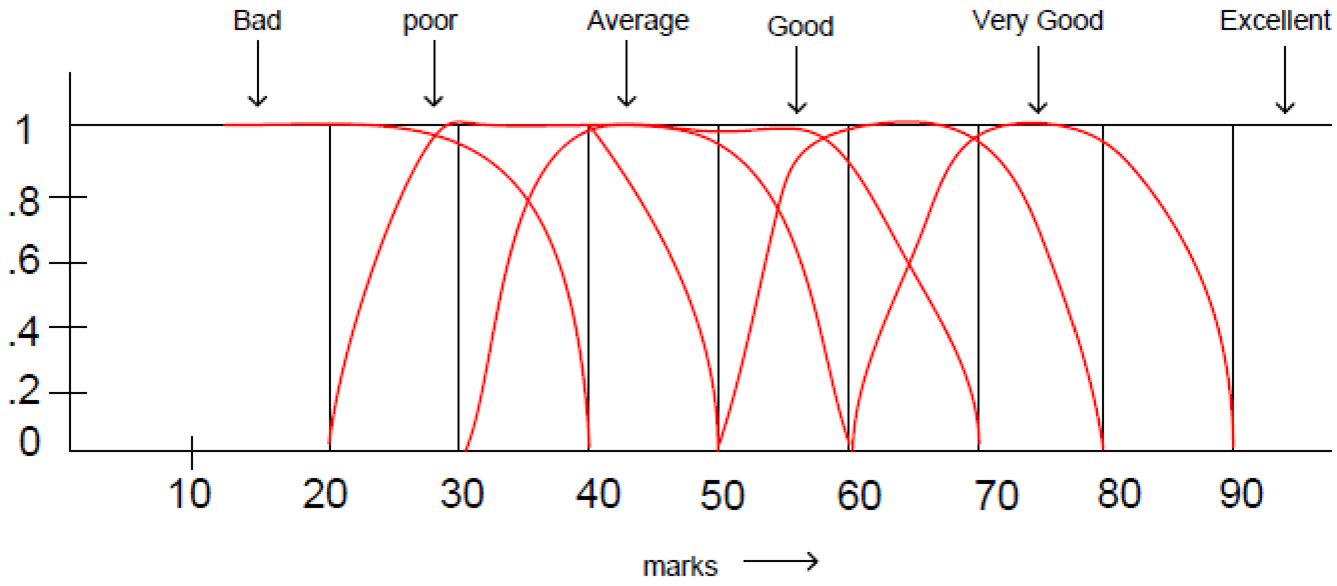
Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35

Membership Functions

A fuzzy implementation will look like the following.

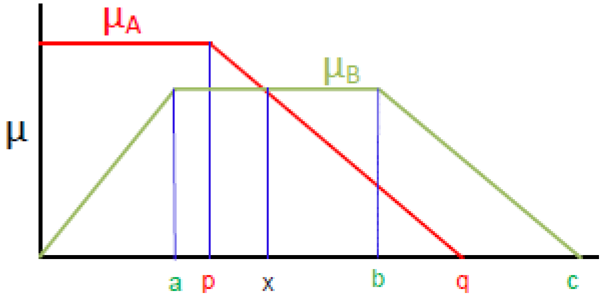


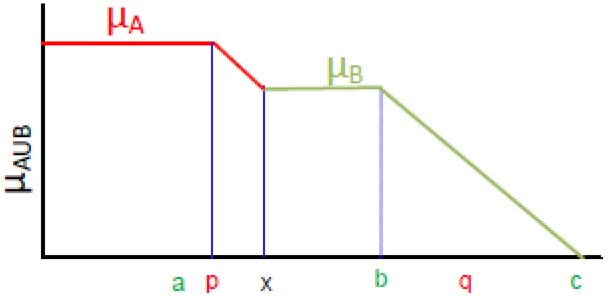
Operations

Union ($A \cup B$):

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



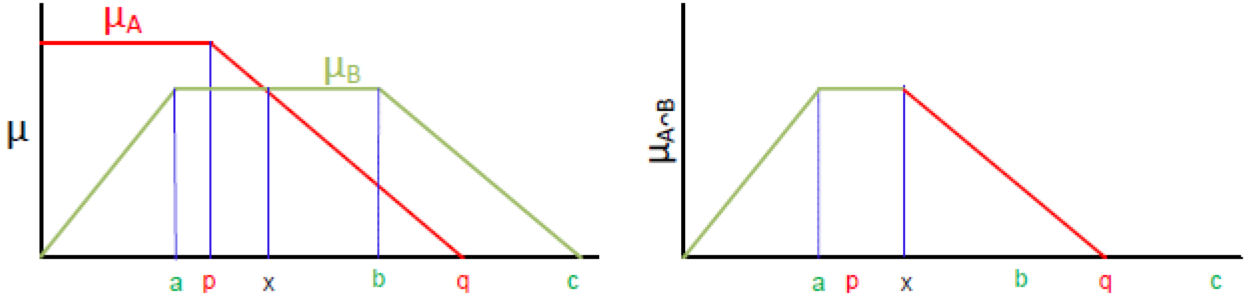


Operations

Intersection ($A \cap B$):

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



Operations

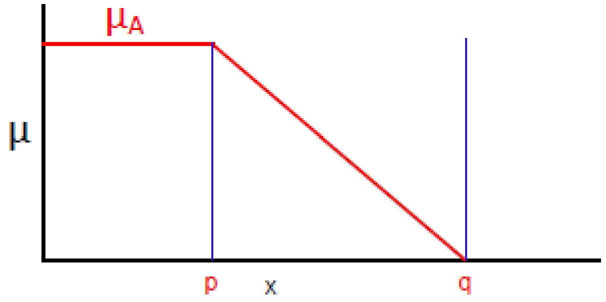
Complement (A^C):

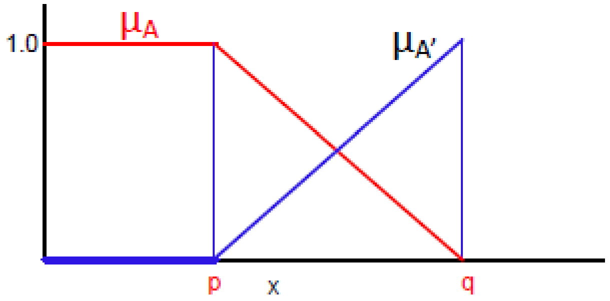
$$\mu_{A_{A^C}}(x) = 1\text{-}\mu_A(x)$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Operations

Algebric product or Vector product (A•B):

$$\mu_{A \bullet B}(x) = \mu_A(x) \bullet \mu_B(x)$$

Scalar product $(\alpha \times A)$:

$$\mu_{\alpha A}(x) = \alpha \cdot \mu_A(x)$$

Operations

Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Difference ($A - B = A \cap B^C$):

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C))$

Bounded Sum: $|\ \mathrm{A(x)} \oplus \mathrm{B(x)}\ |$

$$\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference: $|\ \mathrm{A(x)} \ominus \mathrm{B(x)}\ |$

$$\mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Operations

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = \{\mu_A(x)\}^\alpha$$

- If $\alpha < 1$, then it is called *dilation*
- If $\alpha > 1$, then it is called *concentration*

Operations

Cartesian Product ($A \times B$):

$$\mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$$

Example 3:

A(x) = $\{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$
B(y) = $\{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$\text{A} \times \text{B} = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Operations

Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Operations

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$

Transitivity :

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Operations

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

- **Concentration:**

$$A^k = [\mu_A(x)]^k \ ; k > 1$$

- **Dilation:**

$$A^k = [\mu_A(x)]^k \ ; k < 1$$

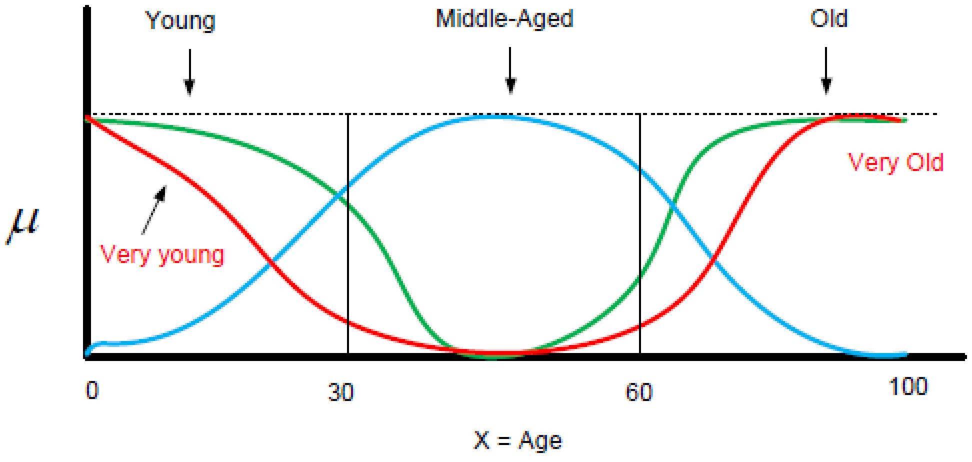
Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.

Similarly, with Old we can have : old, very old, very very old, extremely old etc.
Thus, Extremely old = $(((old)^2)^2)^2$ and so on

Or, More or less old = $A^{0.5} = (old)^{0.5}$

Operations



$$\mu_{young}(x) = bell(x, 20, 2, 0) = \frac{1}{1+(\frac{x}{20})^4}$$

$$\mu_{old}(x) = bell(x, 30, 3, 100) = \frac{1}{1+(\frac{x-100}{30})^6}$$

$$\mu_{middle-aged} = bell(x, 30, 60, 50)$$

Not young = $\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \overline{\mu_{young}(x)}$

Types

- **Ebrahim Mamdani Fuzzy Models**
- **Sugeno Fuzzy Models**
- **Tsukamoto Fuzzy Models**

- The differences between these three FISs lie in the consequents of their fuzzy rules, and thus their aggregation and defuzzification procedures differ accordingly.

Mamdani Fuzzy Model

- The most commonly used fuzzy inference technique is the so-called **Mamdani** method.
- In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination. He applied a set of fuzzy rules supplied by experienced human operators.
- The Mamdani-style fuzzy inference process is performed in four steps:

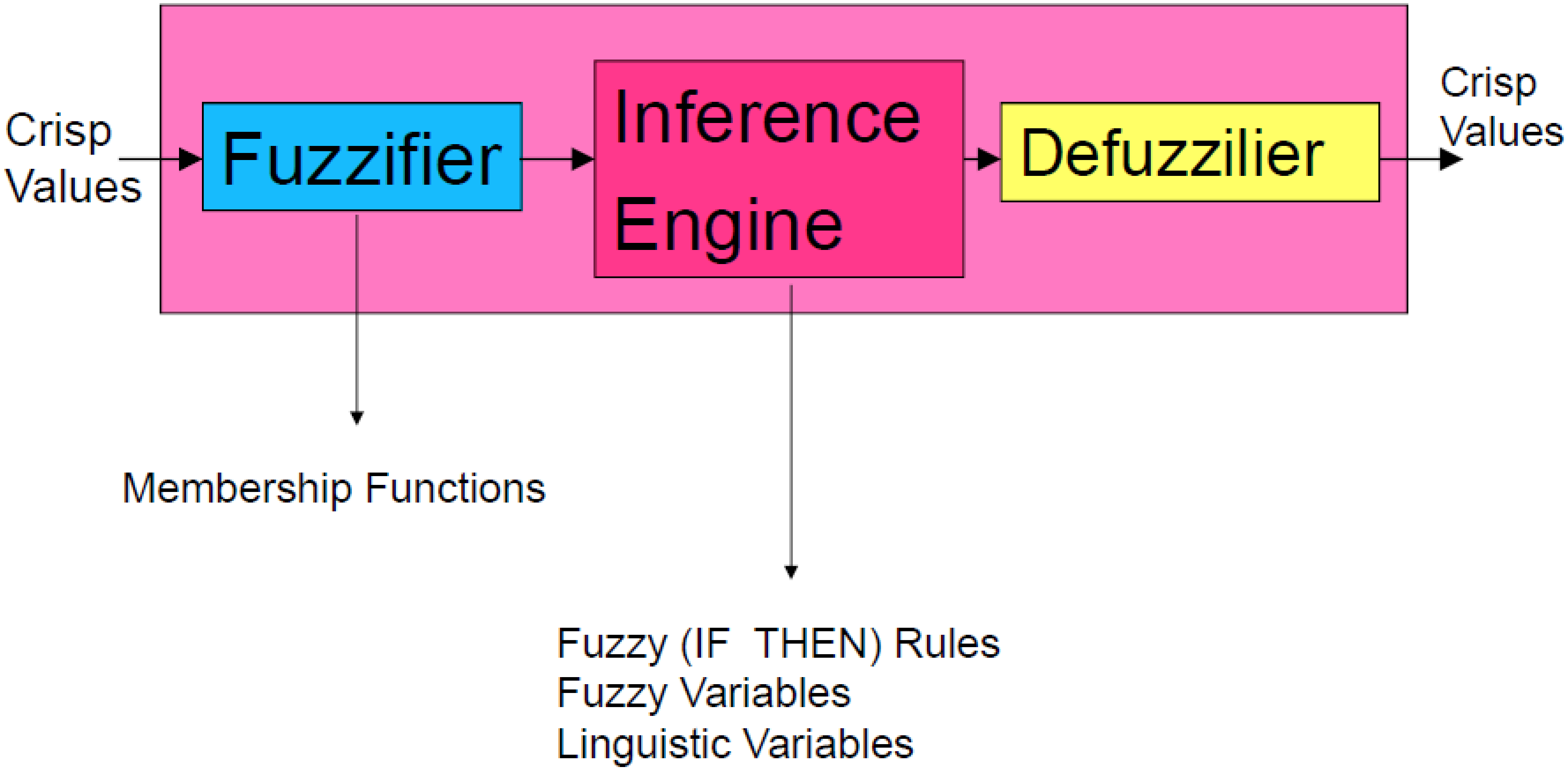
1. Fuzzification of the input variables

2. Rule evaluation (inference)

3. Aggregation of the rule outputs (composition)

4. Defuzzification

Mamdani Fuzzy Model



Crisp Values

Fuzzifier

Inference Engine

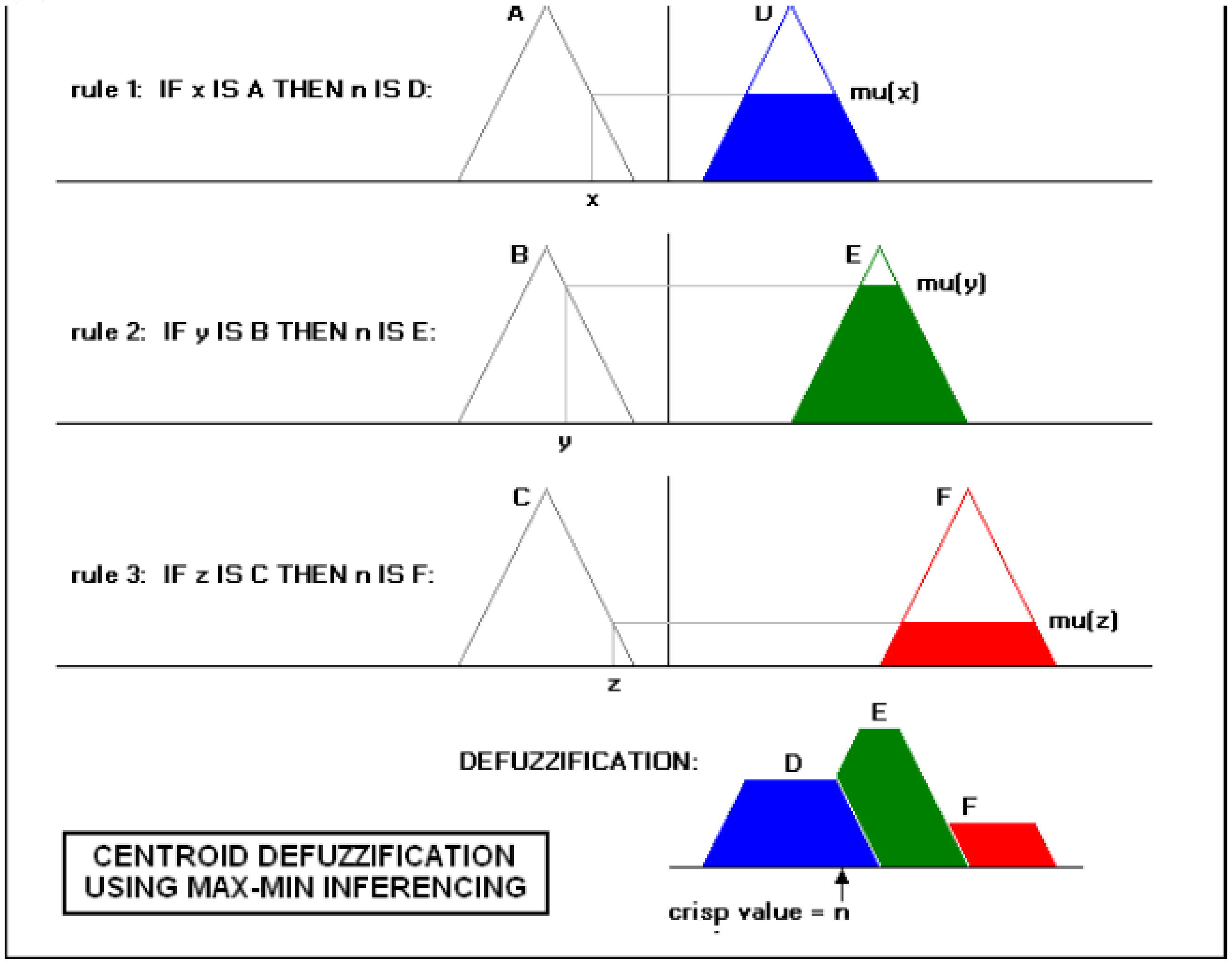
Defuzzilier

Crisp Values

Membership Functions

Fuzzy (IF THEN) Rules
Fuzzy Variables
Linguistic Variables

Mamdani Fuzzy Model

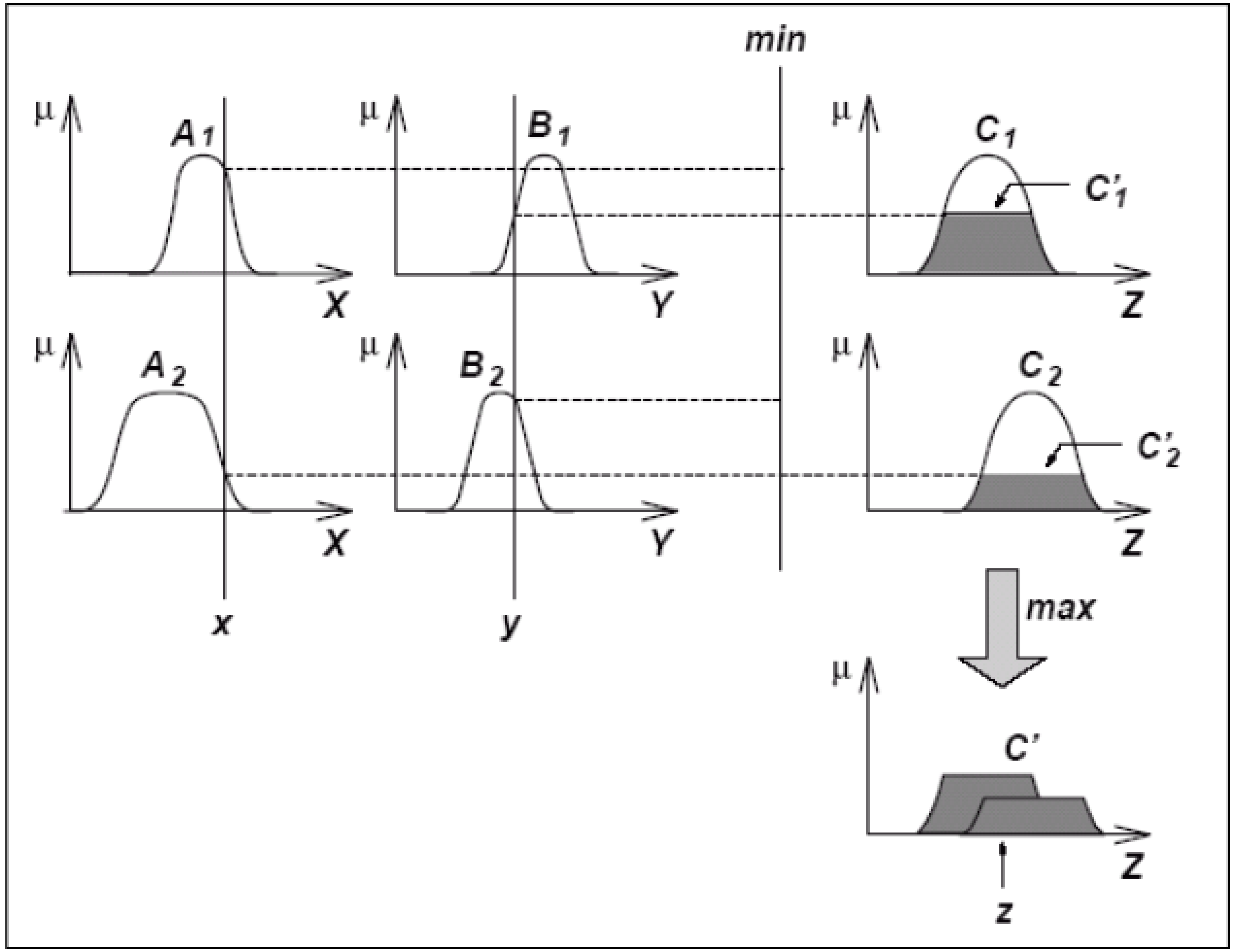


Mamdani composition of three SISO fuzzy outputs

http://en.wikipedia.org/wiki/Fuzzy_control_system

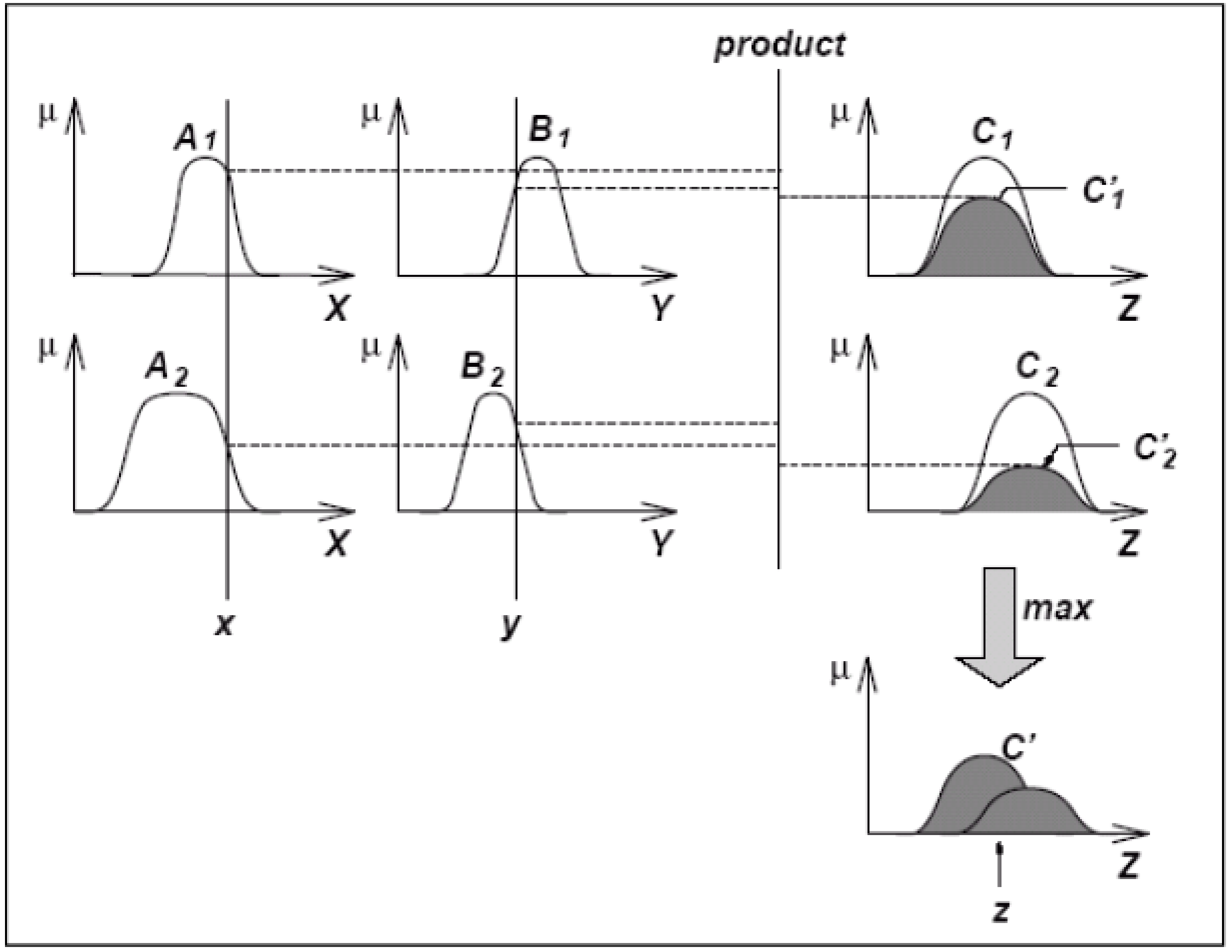
Mamdani Fuzzy Model

The mamdani FIS using **min** and **max** for **T-norm** and **S-norm** and subject to two crisp inputs x and y

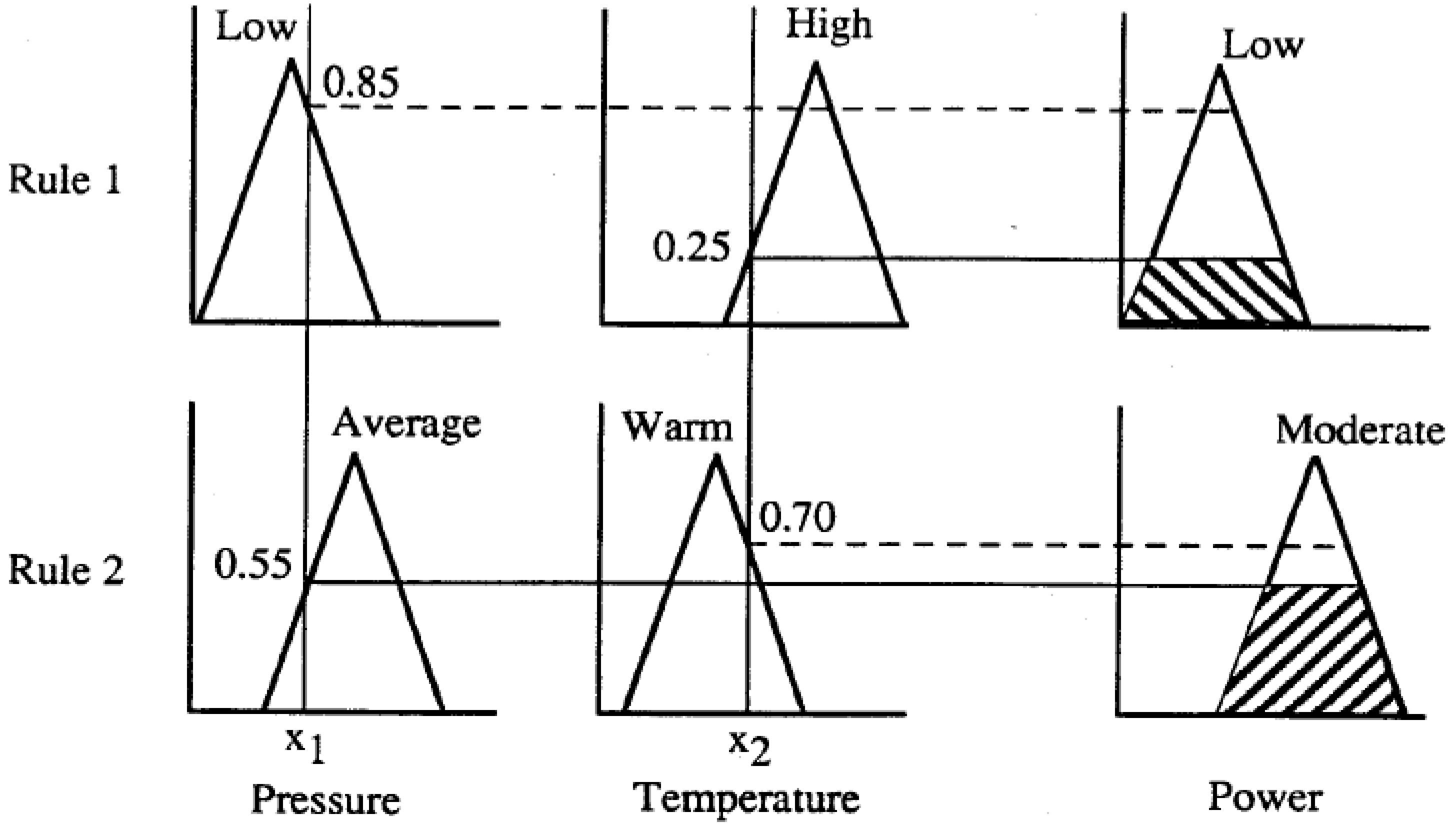


Mamdani Fuzzy Model

The mamdani FIS using **product** and **max** for **T-norm** and **S-norm** and subject to two crisp inputs x and y



Mamdani Fuzzy Model



Rule 1: If pressure is low and temperature is high then power is low
Rule 2: If pressure is average and temperature is warm then power is moderate

Mamdani Fuzzy Model

Two-input, one-ouput example:

If x is A_i and y is B_k then z is $C_{m(i,k)}$

y

x

	B_1	B_2	B_3	B_4
A_1	\	C_2	C_3	C_4
A_2	C_5	C_6	C_7	C_8
A_3	C_9	C_{10}	C_{11}	C_{12}

z

Y

x

	L	**M**	**H**	**VH**
L	Z	L	M	H
M	L	M	H	VH
H	M	H	VH	VH

Playing time

Mamdani Fuzzy Model

- In many applications we have to use crisp values as inputs for controlling of machines and systems.

- So, we have to use a defuzzifier to convert a fuzzy set to a crisp value.

Mamdani Fuzzy Model

- Defuzzification refers to the way a crisp value is extracted from a fuzzy set as a representative value.
- Defuzzification Methods:
    - Centroid of Area
    - Bisector of Area
    - Mean of Max
    - Smallest of Max
    - Largest of Max

Mamdani Fuzzy Model

$$z_{\mathrm{COA}} = \frac{\int_Z \mu_A(z) z \, dz}{\int_Z \mu_A(z) \, dz},$$

- where μ_A is aggregated output MF.
- This is the most widely adopted defuzzification strategy, which is reminiscent of the calculation of expected values of probability distributions.

Mamdani Fuzzy Model

- z_{BOA} satisfies

$$\int_{\alpha}^{z_{\mathrm{BOA}}} \mu_A(z)\,dz = \int_{z_{\mathrm{BOA}}}^{\beta} \mu_A(z)\,dz,$$

$$\boldsymbol{\alpha = min\{z|z \in Z\}} \qquad \boldsymbol{\beta = max\{z|z \in Z\}}$$

- That is, the vertical line $z = z_{BOA}$ partitions the region between $z = \alpha$, $z = \beta$, $y = 0$ and $y = \mu_A(z)$ into two regions with the same area.

Mamdani Fuzzy Model

- z_{MOM} is the mean of maximizing *z* at which the MF reaches maximum μ*. In Symbols,

$$z_{\text{MOM}} = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz},$$

$$\boldsymbol{Z'} = \{\boldsymbol{z} | \boldsymbol{\mu_A}(\boldsymbol{z}) \in \boldsymbol{\mu}^*\}$$

- In particular, if $\mu_A(z)$ has a single maximum at $z = z^*$, then the $z_{MOM} = z^*$.
- Moreover, if $\mu_A(z)$ reaches its maximum whenever

$$z \in [z_{left}, z_{right}]$$

then

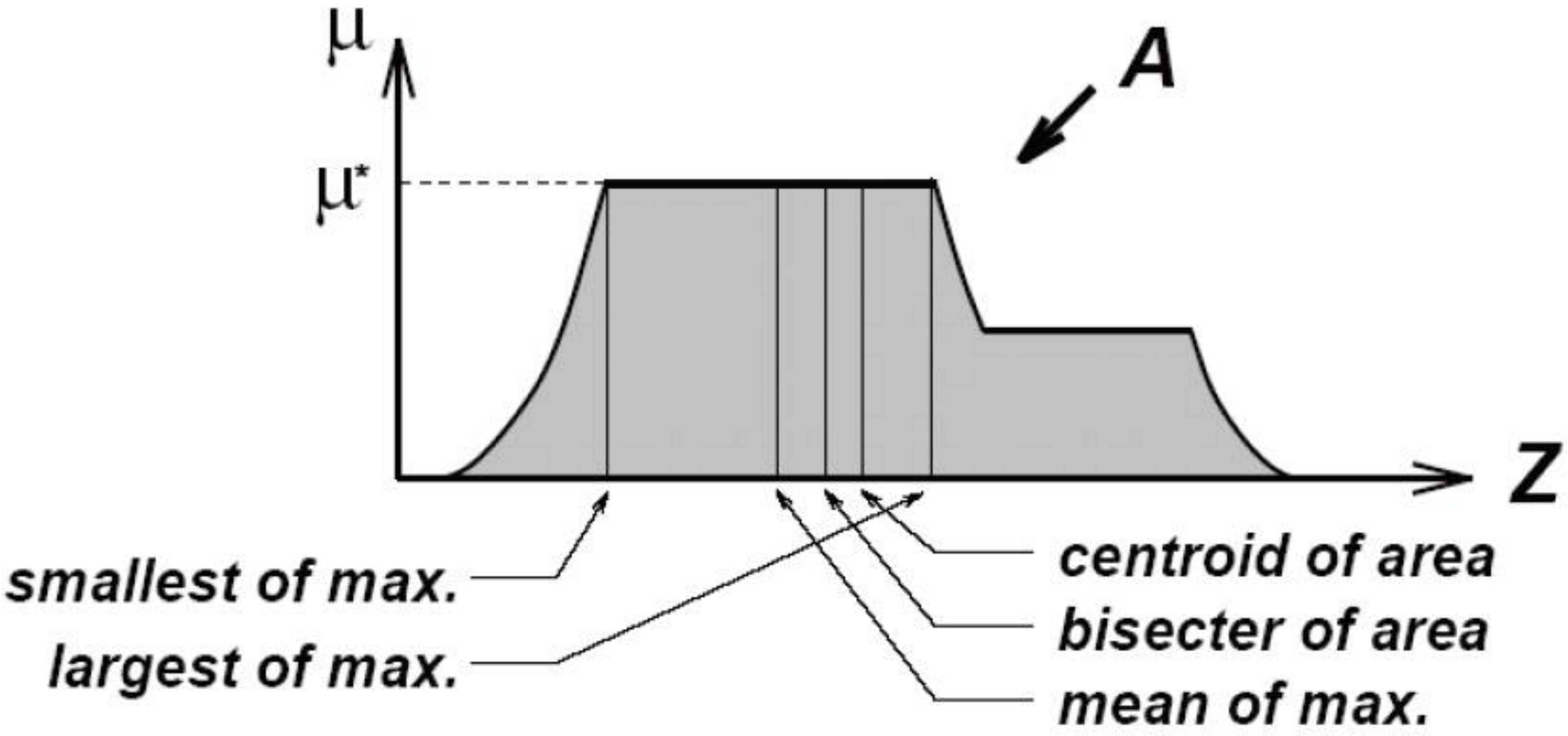
$$z_{MOM} = (z_{left} + z_{right})/2$$

Mamdani Fuzzy Model

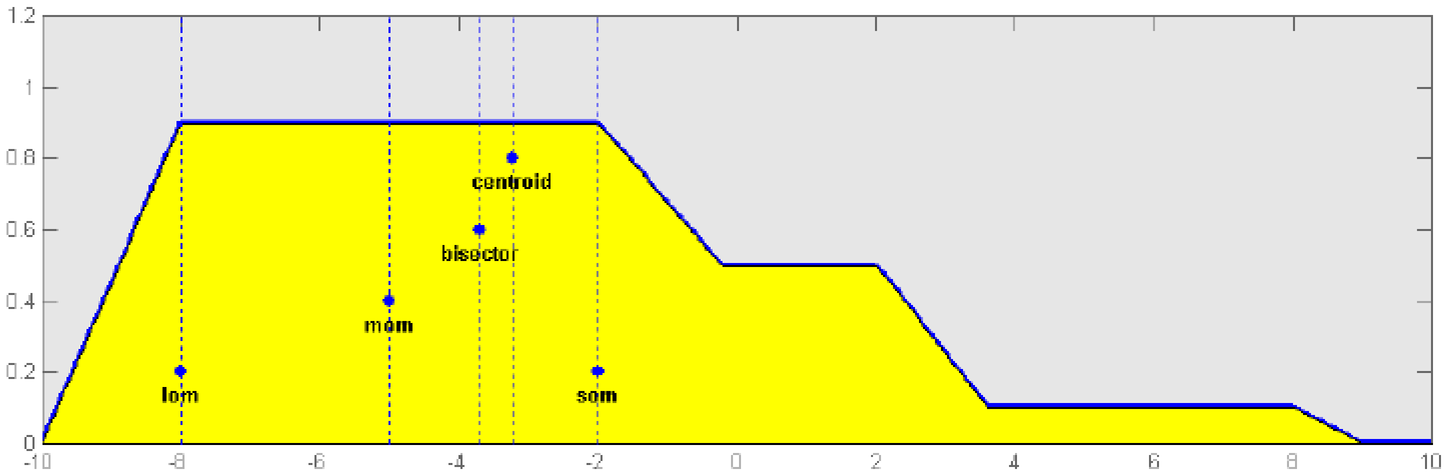
- z_{SOM} is the minimum (in terms of magnitude) of the maximizing z.
- z_{LOM} is the maximum (in terms of magnitude) of the maximizing z.

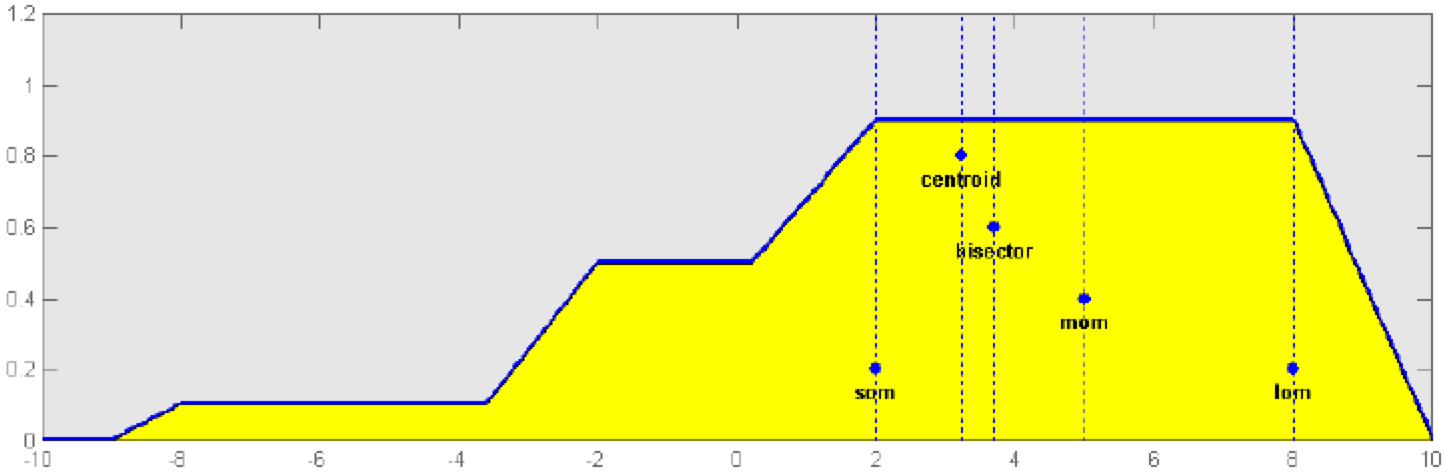
- Because of their obvious bias, z_{SOM} and z_{LOM} are not used as often as the other three defuzzification methods.

Mamdani Fuzzy Model

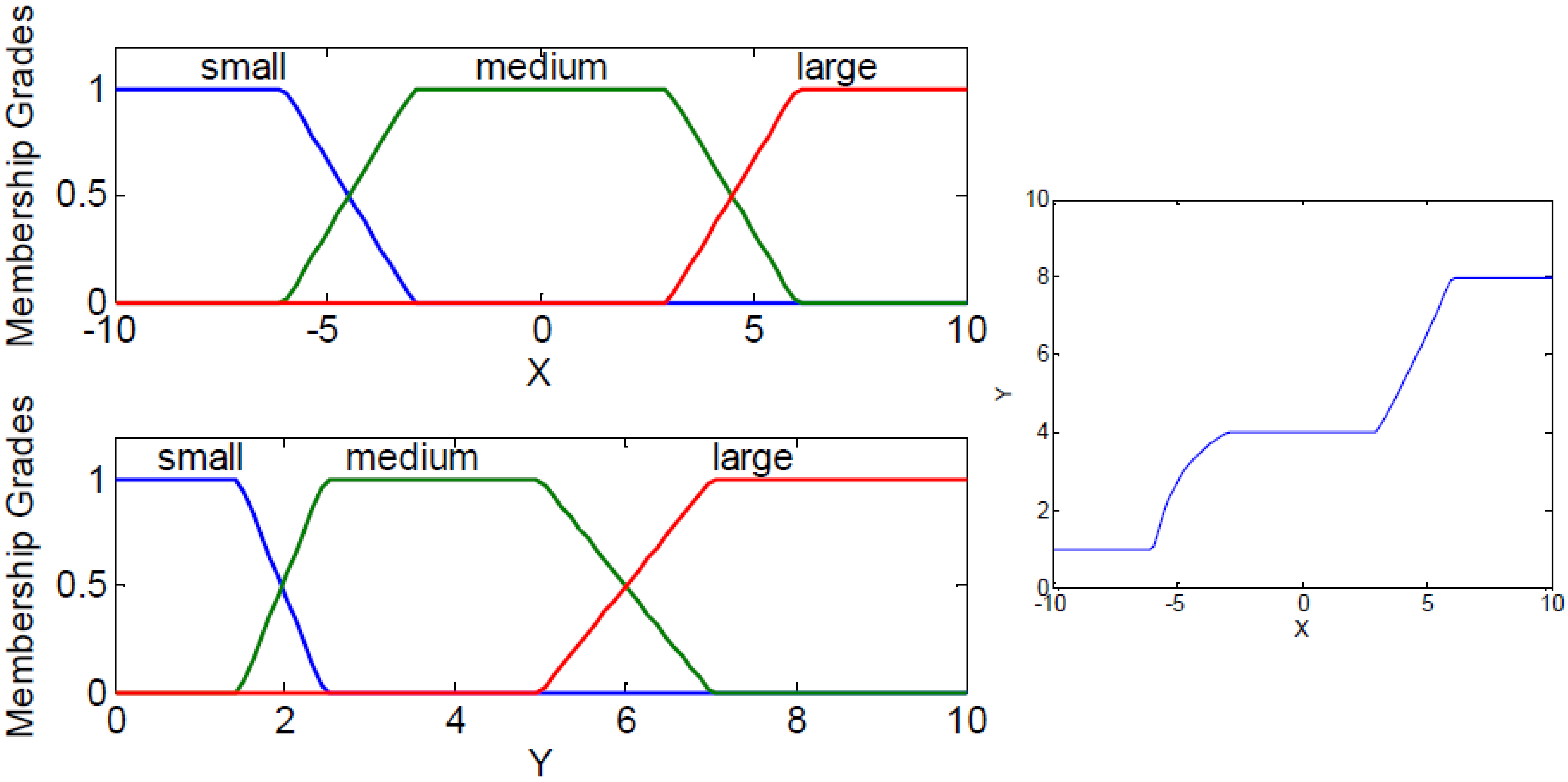


Mamdani Fuzzy Model



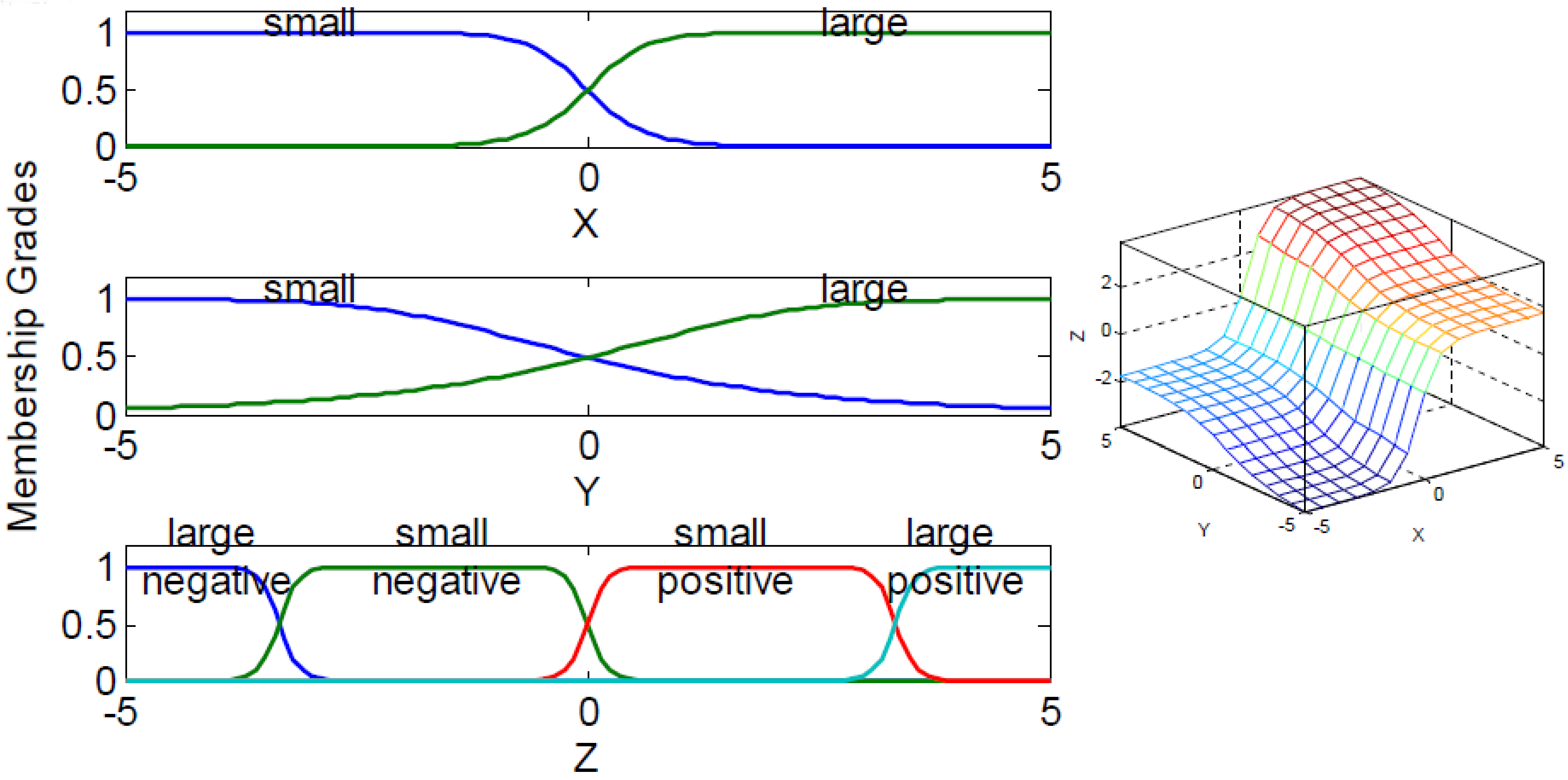


Mamdani Fuzzy Model



a) MFs of the input and output
b) Overall input-output curve

Mamdani Fuzzy Model



a) MFs of the inputs and output
b) Overall input-output curve