Ασκήσεις στην SVD

(1) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ Ποιά η SVD του A?

$$\underset{2\times 2}{U}\cdot\underset{2\times 3}{S}\,\underset{3\times 3}{V^*} = A, \quad \text{όπου } U^*U = UU^* = I_2$$

$\operatorname{rank}(A) = 2 = \min\{2,3\}$ $\qquad V^*V = VV^* = I_3$

• εύρεση ιδιοτιμών του πίνακα $\boxed{A^TA \in \mathbb{R}^{3\times 3}}$

$$A^TA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \text{συμμετρικός}$$

$$\det(A^TA - \lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} =$$

$$= (2-\lambda)\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} + \det\begin{pmatrix} -1 & 0 \\ 1 & 1-\lambda \end{pmatrix} + \det\begin{pmatrix} -1 & 1-\lambda \\ 1 & 0 \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda)^2 - (1-\lambda) - (1-\lambda) = (1-\lambda)\left[(2-\lambda)(1-\lambda) - 2\right]$$

$$= (1-\lambda)(\cancel{2} - 3\lambda + \lambda^2 - \cancel{2}) = \lambda(1-\lambda)(\lambda-3) = -\lambda(\lambda-1)(\lambda-3)$$

Ιδιοτιμές $\sigma(A^TA) = \{3, 1, 0\}$

$$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = 0$$

ιδιάζουσες τιμές του A $\Rightarrow$ $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$

$\sigma_2 = \sqrt{\lambda_2} = 1$

$$\underset{2\times 3}{S} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq 0 \quad !$$

Παρατήρηση $\quad AA^T \in \mathbb{R}^{2\times 2}, \quad A^TA \in \mathbb{R}^{3\times 3}$

Οι πίνακες AA^T και A^TA έχουν τις ίδιες θετικές ιδιοτιμές

- εύρεση του πίνακα $V \in \mathbb{R}^{3\times3}$ ορθομοναδιαίου

$V = [v_1 \ v_2 \ v_3]$ όπου v_1, v_2, v_3 ορθοκανονικά ιδιοδιανύσματα του $A^TA \in \mathbb{R}^{3\times3}$ που αντιστοιχούν στις ιδιοτιμές $\lambda_1, \lambda_2, \lambda_3$

▶ $(A^TA - \lambda_1 I)\hat{v}_1 = 0 \Rightarrow$ για $\lambda_1 = 3 \Rightarrow$

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{aligned} -v_{11} - v_{12} + v_{13} &= 0 \\ -v_{11} - 2v_{12} \quad &= 0 \\ v_{11} \quad -2v_{13} &= 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} -v_{11} - v_{12} + v_{13} &= 0 \\ v_{12} &= -\frac{1}{2}v_{11} \\ v_{13} &= \frac{1}{2}v_{11} \end{aligned}\right\} \Rightarrow \hat{v}_1 = (v_{11}, v_{12}, v_{13}) = \left(v_{11}, \frac{-v_{11}}{2}, \frac{v_{11}}{2}\right)$$

$$= \frac{v_{11}}{2}(2, -1, 1) \quad \forall v_{11} \in \mathbb{R} \setminus \{0\}$$

$\hat{v}_1 = (2, -1, 1)$ ιδιοδιάνυσμα που αντιστοιχεί στην $\lambda_1 = 3$

$$\|\hat{v}_1\| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{2^2 + (-1)^2 + 1} = \sqrt{6}$$

$$v_1 = \frac{\hat{v}_1}{\|\hat{v}_1\|} = \frac{1}{\sqrt{6}}(2, -1, 1) = \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

▶ $(A^TA - \lambda_2 I)\hat{v}_2 = 0 \Rightarrow$ για $\lambda_2 = 1$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = 0 \Rightarrow \left.\begin{aligned} v_{21} - v_{22} + v_{23} &= 0 \\ -v_{21} &= 0 \\ v_{21} &= 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} v_{23} &= v_{22} \\ v_{21} &= 0 \end{aligned}\right\} \Rightarrow \hat{v}_2 = (v_{21}, v_{22}, v_{23}) = (0, v_{22}, v_{22})$$

$$= v_{22}(0, 1, 1) \quad \forall v_{22} \in \mathbb{R} \setminus \{0\}$$

$\hat{v}_2 = (0, 1, 1) \rightsquigarrow$ ιδιοδιάνυσμα που αντιστοιχεί στην $\lambda_2 = 1$

$\|\hat{v}_2\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$ $\quad v_2 = \frac{\hat{v}_2}{\|\hat{v}_2\|} = \frac{1}{\sqrt{2}}(0, 1, 1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\blacktriangleright$ $(A^TA - \lambda_3 I)\hat{v}_3 = 0$ για $\lambda_3 = 0$ $\Rightarrow$

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} 2v_{31} - v_{32} + v_{33} = 0 \\ -v_{31} + v_{32} = 0 \\ v_{31} + v_{33} = 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} v_{32} = v_{31} \\ v_{33} = -v_{31} \end{array}\right\} \Rightarrow \hat{v}_3 = (v_{31}, v_{32}, v_{33}) = (v_{31}, v_{31}, -v_{31}) = v_{31}(1, 1, -1) \quad \forall\, v_{31} \in \mathbb{R} \setminus \{0\}$$

$$\|\hat{v}_3\| = \sqrt{1+1+1} = \sqrt{3}$$

$$v_3 = \frac{\hat{v}_3}{\|\hat{v}_3\|} = \frac{1}{\sqrt{3}}(1, 1, -1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$$

<u>Σημείωση:</u> Τα ιδιοδιανύσματα ενός συμμετρικού πίνακα που αντιστοιχούν σε διαφορετικές ιδιοτιμές είναι μεταξύ τους κάθετα

$$V = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

- εύρεση του πίνακα $U \in \mathbb{R}^{2\times 2}$

$$U = [u_1 \ \ u_2] \qquad u_1 = \frac{1}{\sigma_1} A v_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Σημείωση: A $m\times n$

$r = \operatorname{rank}(A) \leq \min\{m, n\}$

$r = \operatorname{rank}(A) =$ πλήθος των θετικών ιδιαζουσών τιμών του A

(2) $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \in \mathbb{R}^{3\times 2} \rightsquigarrow \operatorname{rank}(A) = 1 < 2 = \min\{2,3\}$

Να βρεθεί η SVD του A.

$A = \underset{3\times 3}{U}\ \underset{3\times 2}{S}\ \underset{2\times 2}{V^*}$

- εύρεση ιδιοτιμών και ιδιοδιανυσμάτων του $A^TA \in \mathbb{R}^{2\times 2}$

$$A^TA = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

$$\det(A^TA - \lambda I) = \det\begin{pmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{pmatrix} = (9-\lambda)^2 - 9^2 = 0$$

$$(9-\lambda)^2 = 9^2 \Rightarrow 9-\lambda = \pm 9 \Rightarrow \lambda \begin{cases} 9+9 = 18 \\ 9-9 = 0 \end{cases}$$

$\lambda_1 = 18$, $\lambda_2 = 0$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{18}$, $\sigma_2 = 0$ $\rightsquigarrow$ ιδιάζουσες τιμές του A

$$\underset{3\times 2}{S} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

για $\lambda_1 = 18 \rightsquigarrow (A^TA - \lambda_1 I)\hat{v}_1 = 0 \Rightarrow \begin{pmatrix} -9 & -9 \\ -9 & -9 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$

$\Rightarrow -9v_{11} - 9v_{12} = 0 \Rightarrow v_{12} = -v_{11}$

$\hat{v}_1 = (v_{11}, v_{12}) = (v_{11}, -v_{11}) = v_{11}(1, -1) \quad \forall v_{11} \in \mathbb{R}\setminus\{0\}$

$\|\hat{v}_1\| = \sqrt{1+1} = \sqrt{2}$

$$\boxed{V_1 = \frac{\hat{v}_1}{\|\hat{v}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)}$$

για $\lambda_2 = 0 \rightsquigarrow (A^TA - \lambda_2 I)\hat{v}_2 = 0 \Rightarrow \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0$

$\Rightarrow 9v_{21} - 9v_{22} = 0 \Rightarrow v_{21} = v_{22}$

$\hat{v}_2 = (v_{21}, v_{22}) = (v_{21}, v_{21}) = v_{21}(1,1) \quad \forall v_{21} \in \mathbb{R} \setminus \{0\}$

$\|\hat{v}_2\| = \sqrt{2} \qquad v_2 = \dfrac{\hat{v}_2}{\|\hat{v}_2\|} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$V = [v_1 \ \ v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \begin{array}{l} v_1 \perp v_2 \\ \|v_1\| = \|v_2\| = 1 \end{array}$

- εύρεση του ορθομοναδιαίου πίνακα $U \in \mathbb{R}^{3\times 3}$

$U = [u_1 \ \ u_2 \ \ u_3]$

- $u_1 = \dfrac{1}{\sigma_1} A v_1 = \dfrac{1}{\sqrt{18}} \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

- Τα διανύσματα $\{u_1, u_2, u_3\}$ αποτελούν ορθοκανονική βάση του $\mathbb{R}^3$ (δηλ. $\langle u_1, u_2\rangle = \langle u_2, u_3\rangle = \langle u_1, u_3\rangle = 0$ και παράγουν τον $\mathbb{R}^3$)

$\langle u_2, u_1 \rangle = 0 \Rightarrow \left\langle (u_{21}, u_{22}, u_{23}), \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)\right\rangle = 0$

$\dfrac{u_{21}}{3} - \dfrac{2u_{22}}{3} + \dfrac{2u_{23}}{3} = 0 \Rightarrow u_{21} - 2u_{22} + 2u_{23} = 0$

$u_{21} = 2u_{22} - 2u_{23}$

$u_2 = (u_{21}, u_{22}, u_{23}) = (2u_{22} - 2u_{23}, u_{22}, u_{23}) =$

$= (2u_{22}, u_{22}, 0) + (-2u_{23}, 0, u_{23})$

$= u_{22}\underbrace{(2,1,0)}_{w_1} + u_{23}\underbrace{(-2,0,1)}_{w_2}$

Άρα $w_1 = (2,1,0)$ και $w_2 = (-2,0,1)$

είναι κάθετα στο $\boxed{u_1 = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)}$

$w_1 \perp u_1$ ✓

$w_2 \perp u_1$ ✓

$w_1 \not\perp w_2$? $\langle w_1, w_2 \rangle = -4 \neq 0$

Εφαρμόζουμε τη μέθοδο Gram-Schmidt

για να ορθογωνιοποιήσουμε τα w_1, w_2 :

$$\hat{u}_2 = w_1 = (2,1,0)$$

$$\hat{u}_3 = w_2 - \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1 =$$

$$= (-2,0,1) - \frac{-2\cdot 2 + 0\cdot 1 + 1\cdot 0}{2^2 + 1^2 + 0^2}(2,1,0)$$

$$= (-2,0,1) - \frac{-4}{5}(2,1,0)$$

$$= (-2,0,1) + \left(\frac{8}{5}, \frac{4}{5}, 0\right) = \left(\frac{-2}{5}, \frac{4}{5}, 1\right)$$

$$u_2 = \frac{\hat{u}_2}{\|\hat{u}_2\|} = \frac{1}{\sqrt{5}}(2,1,0) = \boxed{\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)} \rightsquigarrow u_2$$

$$u_3 = \frac{\hat{u}_3}{\|\hat{u}_3\|} = \frac{5}{\sqrt{45}}\left(\frac{-2}{5}, \frac{4}{5}, 1\right) = \boxed{\left(\frac{-2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, \frac{5}{\sqrt{45}}\right)} \to u_3$$

$$\|\hat{u}_3\| = \sqrt{\frac{4}{25} + \frac{16}{25} + 1} = \sqrt{\frac{45}{25}} = \frac{\sqrt{45}}{5}$$

$$U = [u_1\ u_2\ u_3] = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$A = USV^*$$

Θεμελιώδεις υπόχωροι του $A_{m\times n}$

$$\underset{m\times n}{A} : \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m}$$

- Χώρος στηλών του $A = C(A)$

  $C(A) = \{ x \in \mathbb{R}^m : x \text{ στήλη του } A \}$

- Μηδενόχωρος του $A = N(A)$

  $N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

- Χώρος γραμμών του $A = C(A^T)$

  $C(A^T) = \{ x \in \mathbb{R}^n : x \text{ γραμμή του } A$

- Μηδενόχωρος του $A^T = N(A^T)$

  $N(A^T) = \{ x \in \mathbb{R}^m : A^T x = 0 \}$

$$\underset{m\times n}{A} = U \cdot S \cdot V^T$$

$$= \left[u_1 \cdots u_r \; u_{r+1} \cdots u_m \right] \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots \sigma_r & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$r = \text{rank}(A) =$ πλήθος θετικών ιδιαζουσών τιμών

$u_1 \cdots u_r$: ορθ. βάση του $C(A)$

$u_{r+1} \cdots u_m$: ορθοκ. βάση του $N(A^T)$

$v_1^T \cdots v_r^T$: ορθοκ. βάση $C(A^T)$

$v_{r+1}^T \cdots v_n^T$: ορθ. βάση $N(A)$

$C(A) \perp N(A^T)$

$C(A^T) \perp N(A)$