Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 426 (2007) 841–854
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Improving bounds for eigenvalues of complex matrices using traces ☆

Ting-Zhu Huang [a,*], Lin Wang [b]

[a] *School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China*
[b] *School of Mathematics, Beijing University, Beijing, PR China*

Received 8 January 2007; accepted 11 June 2007
Available online 16 June 2007
Submitted by R.A. Brualdi

Abstract

Let A be a complex matrix of order n with eigenvalues λ_j $(j = 1, 2, \ldots, n)$ and m be any integer satisfying rank $A \leqslant m \leqslant n$. The bound for $\sum |\lambda_j|^2$ by Kress, de Vries, and Wegmann is strengthened. Furthermore, new bounds are presented to estimate the spectral radius of A using m and traces of A, A^2, A^*A and $A^*A - AA^*$. We also improve some Wolkowicz–Styan bounds and previous localization of eigenvalues in rectangular or elliptic regions using traces. Several simple lower bounds for the spectral radius are given, involving tr A, tr A^2, tr A^3, and m.

AMS classification: 15A42; 15A18

Keywords: Complex matrix; Eigenvalue; Spectral radius; Localization

1. Introduction

Traces of matrices are used in many results on localization of eigenvalues, e.g. [1–4]. Let A be a complex matrix of order n with eigenvalues λ_j $(j = 1, 2, \ldots, n)$. Use $\|\cdot\|$ as the Euclidean norm throughout.

☆ Supported by NCET of Chinese ministry of education (NCET-04-0893), and project of academic leader and group of UESTC.
* Corresponding author. Tel.: +86 2883201175; fax: +86 2883200131.
E-mail addresses: tzhuang@uestc.edu.cn (T.-Z. Huang), linwang@math.pku.edu.cn (L. Wang).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2007.06.008

Eberlein [5] showed that

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant K = \|A\|^2 - \frac{\|A^*A - AA^*\|^2}{6\|A\|^2},$$

while Kress et al. [6] gave a tighter bound:

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant K_A = \left(\|A\|^4 - \frac{1}{2}\|A^*A - AA^*\|^2\right)^{1/2}.$$

Wolkowicz and Styan [2] adopted K_A, K_B, K_C as upper bounds respectively for $\sum_{j=1}^{n} |\lambda_j|^2$, $\sum_{j=1}^{n} (\Re\lambda_j)^2$, $\sum_{j=1}^{n} (\Im\lambda_j)^2$, where

$$K_B = \begin{cases} \frac{1}{2}((\|A\|^2 + \Re\mathrm{tr}\, A^2)^2 - \frac{1}{2}\|A^*A - AA^*\|^2)^{1/2} & \text{if } \Re\mathrm{tr}\, A^2 \geqslant 0, \\ \frac{1}{2}(\|A\|^2 + \Re\mathrm{tr} A^2) - \frac{\|A^*A - AA^*\|^2}{12\|A\|^2} & \text{otherwise,} \end{cases}$$

$$K_C = \begin{cases} \frac{1}{2}((\|A\|^2 - \Re\mathrm{tr}\, A^2)^2 - \frac{1}{2}\|A^*A - AA^*\|^2)^{1/2} & \text{if } \Re\mathrm{tr}\, A^2 \leqslant 0, \\ \frac{1}{2}(\|A\|^2 - \Re\mathrm{tr}\, A^2) - \frac{\|A^*A - AA^*\|^2}{12\|A\|^2} & \text{otherwise.} \end{cases}$$

O. Rojo et al. applied in [3]

$$\sum_{j=1}^{n} (\Re\lambda_j)^2 \leqslant \frac{1}{2}(K + \Re\mathrm{tr}\, A^2), \tag{1a}$$

$$\sum_{j=1}^{n} (\Im\lambda_j)^2 \leqslant \frac{1}{2}(K - \Re\mathrm{tr}\, A^2), \tag{1b}$$

while they used the two inequalities below in [4]:

$$\sum_{j=1}^{n} (\Re\lambda_j)^2 \leqslant \frac{1}{2}(K_A + \Re\mathrm{tr}\, A^2), \tag{2a}$$

$$\sum_{j=1}^{n} (\Im\lambda_j)^2 \leqslant \frac{1}{2}(K_A - \Re\mathrm{tr}\, A^2). \tag{2b}$$

In Section 2, we give an upper bound for $\sum |\lambda_j|^2$ which is tighter than K_A. Applying the tighter bound, we develop upper bounds for the spectral radius, and improve some results in [1–4] to localize the spectrum of a matrix. We also present some simple lower bounds for the spectral radius in Section 3. The paper ends with several examples in Section 4.

2. Bounds for eigenvalues

We first present a new upper bound for $\sum_{j=1}^{n} |\lambda_j|^2$. Define

$$c(A) = \left(\left(\|A\|^2 - \frac{|\mathrm{tr}\, A|^2}{n}\right)^2 - \frac{1}{2}\|A^*A - AA^*\|^2\right)^{1/2} + \frac{|\mathrm{tr}\, A|^2}{n}. \tag{3}$$

Lemma 2.1. *Let A be a complex matrix of order n with eigenvalues λ_j $(j = 1, 2, \ldots, n)$. Then*

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant c(A), \tag{4}$$

$$\sum_{j=1}^{n}(\Re\lambda_j)^2 \leqslant \frac{1}{2}(c(A)+\Re\mathrm{tr}\, A^2), \tag{5a}$$

$$\sum_{j=1}^{m}(\Im\lambda_j)^2 \leqslant \frac{1}{2}(c(A)-\Re\mathrm{tr}\, A^2). \tag{5b}$$

Proof. Let

$$B = A - \frac{\mathrm{tr}\, A}{n} I,$$

where I is the identity matrix of order n. Then $\lambda_j - \frac{\mathrm{tr}\, A}{n}$ $(j = 1, 2, \ldots, n)$ are eigenvalues of B. By Kress et al. [5], we have

$$\sum_{j=1}^{n}\left|\lambda_j - \frac{\mathrm{tr}\, A}{n}\right|^2 = \sum_{j=1}^{n}|\lambda_j|^2 - \frac{|\mathrm{tr}\, A|^2}{n} \leqslant \left(\|B\|^4 - \frac{1}{2}\|B^*B - BB^*\|^2\right)^{1/2}$$
$$= c(A) - \frac{|\mathrm{tr}\, A|^2}{n}.$$

Thus, the inequality (4) holds. Since

$$\sum_{j=1}^{n}(\Re\lambda_j)^2 + \sum_{j=1}^{n}(\Im\lambda_j)^2 = \sum_{j=1}^{n}|\lambda_j|^2,$$
$$\sum_{j=1}^{n}(\Re\lambda_j)^2 - \sum_{j=1}^{n}(\Im\lambda_j)^2 = \Re\mathrm{tr}\, A^2,$$

the inequalities (5a) and (5b) follow from (4). □

Lemma 2.2. *Let A be a complex matrix of order n. Then*

$$c(A) \leqslant K_A \leqslant K, \tag{6}$$

$$\frac{1}{2}(c(A)+\Re\mathrm{tr}\, A^2) \leqslant K_B, \tag{7a}$$

$$\frac{1}{2}(c(A)-\Re\mathrm{tr}\, A^2) \leqslant K_C. \tag{7b}$$

Equality holds on the left of (6) if and only if A is normal or $\mathrm{tr}\, A = 0$. *Equality holds on the right of (6) if and only if A is normal.*

Proof

$$c(A) \leqslant K_A$$
$$\Longleftrightarrow c^2(A) \leqslant \|A\|^4 - \frac{1}{2}\|A^*A - AA^*\|^2 \quad \text{i.e. } \|A\|^2 \geqslant c(A) \text{ or tr } A = 0$$
$$\Longleftrightarrow \|A^*A - AA^*\| \geqslant 0 \quad \text{or tr } A = 0$$
$$K_A \leqslant K$$

$$
\begin{aligned}
&\iff \|A\|^4 - \frac{1}{2}\|A^*A - AA^*\|^2 \leqslant \left(\|A\|^2 - \frac{\|A^*A - AA^*\|^2}{6\|A\|^2}\right)^2 \\
&\iff \frac{\|A^*A - AA^*\|^2}{6} + \frac{\|A^*A - AA^*\|^4}{36\|A\|^4} \geqslant 0 \\
&\iff \|A^*A - AA^*\| \geqslant 0.
\end{aligned}
$$

To prove the inequality (7a), it is sufficient to prove

$$\frac{1}{2}(K_A + \Re\mathrm{tr}\, A^2) \leqslant K_B. \tag{8}$$

If $\Re\mathrm{tr}\, A^2 < 0$, the right side of (6) implies (8). If $\Re\mathrm{tr}\, A^2 = 0$, the inequality (8) collapses to equality. Let x, y, z respectively denote $\|A\|^2$, $\Re\mathrm{tr}\, A^2$, $\|A^*A - AA^*\|/\sqrt{2}$ here. If $\Re\mathrm{tr}\, A^2 > 0$, the inequality (8) holds if and only if

$$
\begin{aligned}
&\sqrt{x^2 - z^2} + y \leqslant \sqrt{(x+y)^2 - z^2} \quad \text{if and only if} \\
&(\sqrt{x^2 - z^2} + y)^2 \leqslant (x+y)^2 - z^2 \quad \text{i.e.} \\
&\sqrt{x^2 - z^2} \leqslant x \quad \text{if and only if} \\
&z^2 \geqslant 0.
\end{aligned}
$$

We omit the proof of (7b), which is similar to (7a). □

All equalities in Lemma 2.2 hold if A is normal.

Then our bound (4) strengthen bounds for $\sum|\lambda_j|^2$ by [5,6]. Lemma 2.2 also implies that our upper bounds by (5) perform better than (1) and (2).

Following the idea of [1, Lemma 2.1, Lemma 2.2], we have

Lemma 2.3. *If real numbers x_j $(j = 1, 2, \ldots, m)$ satisfy $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m$, then, for $1 \leqslant k \leqslant l \leqslant m$,*

$$\frac{1}{l-k+1}\sum_{j=k}^{l} x_j \leqslant \frac{\sum_{j=1}^m x_j}{m} + \left(\frac{m-l}{ml}\right)^{1/2}\left(\sum_{j=1}^m x_j^2 - \frac{\left(\sum_{j=1}^m x_j\right)^2}{m}\right)^{1/2}, \tag{9}$$

$$\left|x_k - \frac{\sum_{j=1}^m x_j}{m}\right| \leqslant \left(\frac{m-1}{m}\right)^{1/2}\left(\sum_{j=1}^m x_j^2 - \frac{\left(\sum_{j=1}^m x_j\right)^2}{m}\right)^{1/2}. \tag{10}$$

Proof. Let $\boldsymbol{e}_j$ $(j = 1, 2, \ldots, m)$ be the standard basis of the linear space $\mathbb{R}^m$. Let

$$\boldsymbol{x} = \sum_{j=1}^m x_j \boldsymbol{e}_j, \quad \boldsymbol{y} = \frac{1}{l-k+1}\sum_{j=k}^{l}\boldsymbol{e}_j, \quad C = I - \left(\sum_{j=1}^m \boldsymbol{e}_j\right)\left(\sum_{j=1}^m \boldsymbol{e}_j^T\right)\Big/ m,$$

where I is the identity matrix of order m.

Since C is positive semidefinite, it holds that

$$(\boldsymbol{x} - \lambda \boldsymbol{y})^T C(\boldsymbol{x} - \lambda \boldsymbol{y}) = \boldsymbol{x}^T C\boldsymbol{x} - 2\lambda \boldsymbol{y}^T C\boldsymbol{x} + \lambda^2 \boldsymbol{y}^T C\boldsymbol{y} \geqslant 0.$$

Take $\lambda = \frac{\boldsymbol{y}^T C\boldsymbol{x}}{\boldsymbol{y}^T C\boldsymbol{y}}$, yielding

$$|\boldsymbol{y}^T C\boldsymbol{x}| \leqslant \sqrt{\boldsymbol{y}^T C\boldsymbol{y}}\sqrt{\boldsymbol{x}^T C\boldsymbol{x}}. \tag{11}$$

Take $k = 1$ in (11) and use

$$\frac{1}{l-k+1}\sum_{j=k}^{l} x_j \leqslant \frac{1}{l}\sum_{j=1}^{l} x_j,$$

then we obtain (9).

Take $l = k$ in (11), that is (10). $\square$

Now Lemmas 2.1 and 2.3 can work together to estimate the spectral radius. Define

$$f_A(\theta) = \frac{\Re(\mathrm{e}^{\mathrm{i}\theta}\mathrm{tr}\, A)}{m} + \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \Re\left(\mathrm{e}^{2\mathrm{i}\theta}\left(\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right)\right)\right)^{1/2}.$$

Theorem 2.1. *Let A be a complex matrix of order n with eigenvalues λ_j $(j = 1, 2, \ldots, n)$ satisfying $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$, and let m be any integer such that* $\mathrm{rank}(A) \leqslant m \leqslant n$. *Then for any $1 \leqslant k \leqslant m$,*

$$\left|\lambda_k - \frac{\mathrm{tr}\, A}{m}\right| \leqslant \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \left|\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right|\right)^{1/2}. \tag{12}$$

$$\rho(A) \leqslant \max_{\theta} f_A(\theta). \tag{13}$$

Proof. Notice $|\lambda_j| = 0 (j > m)$. Observe that

$$\begin{aligned}
&2\left(\sum_{j=1}^{m}(\Re(\mathrm{e}^{\mathrm{i}\theta}\lambda_j))^2 - \left(\sum_{j=1}^{m}\Re(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)\right)^2 \Big/ m\right)\\
&= \sum_{j=1}^{m}|\lambda_j|^2 + \Re\left(\mathrm{e}^{2\mathrm{i}\theta}\sum_{j=1}^{m}\lambda_j^2\right) - \frac{1}{m}\left|\sum_{j=1}^{m}\lambda_j\right|^2 - \frac{1}{m}\Re\left(\mathrm{e}^{2\mathrm{i}\theta}\left(\sum_{j=1}^{m}\lambda_j\right)^2\right)\\
&\leqslant c(A) + \Re(\mathrm{e}^{2\mathrm{i}\theta}\mathrm{tr} A^2) - \frac{1}{m}|\mathrm{tr}\, A|^2 - \frac{1}{m}\Re(\mathrm{e}^{2\mathrm{i}\theta}\mathrm{tr}^2 A)\\
&\leqslant c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \left|\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right|.
\end{aligned} \tag{14}$$

Given (14), applying (10) to $\Re(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)$ $(j = 1, \ldots, m)$, where $\theta = -\arg\left(\lambda_k - \frac{\mathrm{tr}\, A}{m}\right)$, yields (12).

Applying (14) without its last line, and substituting $\Re(\mathrm{e}^{\mathrm{i}\theta}\lambda_j)(j = 1, \ldots, m)$ for x_j $(j = 1, 2, \ldots, n)$ in (10), where $\theta = -\arg\lambda_k$, we obtain

$$|\lambda_k| \leqslant f_A(\theta).$$

Since $\rho(A) = \max_k |\lambda_k|$, the inequality (13) holds. Existence of the maximum relies on the evidence that the function $f_A(\theta)$ is periodic and continuous over its real domain. $\square$

The inequality (13) at once gives a simple upper bound for the spectral radius.

Corollary 2.1. *Let A be a complex matrix of order n with m defined as in Theorem* 2.1. *Then*

$$\rho(A) \leqslant \frac{|\mathrm{tr}\, A|}{m} + \left(\frac{m-1}{2m}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \left|\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right|\right)^{1/2}. \tag{15}$$

Given the conditions in Corollary 2.1, it can be concluded that

$$\left|\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right| = \left|\sum_{j=1}^{m}\left(\lambda_j - \frac{\sum_{j=1}^{m}\lambda_j}{m}\right)^2\right| \leqslant \sum_{j=1}^{m}|\lambda_j|^2 - \frac{\left|\sum_{j=1}^{m}\lambda_j\right|^2}{m} \leqslant c(A) - \frac{|\mathrm{tr}\, A|^2}{m}. \tag{16}$$

Due to Lemma 2.1 and (16), our estimate (15), when $m = n$, improves the bound below by [1, Theorem 3.1, (3.12)]:

$$\rho(A) \leqslant \frac{|\mathrm{tr} A|}{n} + \left(\frac{n-1}{n}\right)^{1/2} \left(\|A\|^2 - \frac{|\mathrm{tr}\, A|^2}{n}\right)^{1/2}. \tag{17}$$

Moreover, there exist a series of theoretical upper bounds approaching to $\rho(A)$.

Theorem 2.2. *Let A be a complex matrix of order n with m defined as in Theorem* 2.1. *Define matrix functions*

$$f_1(A) = \frac{|\mathrm{tr}\, A|}{m} + \left(\frac{m-1}{2m}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \left|\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right|\right)^{1/2},$$

$$f_2(A) = \frac{|\mathrm{tr} A|}{n} + \left(\frac{n-1}{n}\right)^{1/2} \left(\|A\|^2 - \frac{|\mathrm{tr}\, A|^2}{n}\right)^{1/2}.$$

Then

$$\lim_{t\to+\infty}(f_1(A^t))^{1/t} = \rho(A) = \lim_{t\to+\infty}(f_2(A^t))^{1/t}. \tag{18}$$

Proof. Since $c(A) \leqslant \|A\|^2$ and $|\mathrm{tr}\, A| \leqslant \sqrt{mc(A)} \leqslant \sqrt{m}\|A\|$ by (16), the inequalities (15) and (16) yield

$$\begin{aligned}
\rho(A) &= (\rho(A^t))^{1/t} \\
&\leqslant (f_1(A^t))^{1/t} \\
&\leqslant \left(\frac{1}{\sqrt{m}}\|A^t\| + \left(\frac{m-1}{m}\right)^{1/2}\left(c(A^t) - \frac{|\mathrm{tr}\, A^t|^2}{m}\right)^{1/2}\right)^{1/t}
\end{aligned}$$

$$\leqslant \left(\frac{1}{\sqrt{m}}\|A^t\| + \left(\frac{m-1}{m}\right)^{1/2}\|A^t\|\right)^{1/t}$$

$$\leqslant \left(\frac{1+\sqrt{m-1}}{\sqrt{m}}\right)^{1/t}\|A^t\|^{1/t}.$$

Since $\lim_{t\to+\infty}\|A^t\|^{1/t} = \rho(A)$ as Gelfand's theorem in [7, p. 299], the left side of (18) is true. The proof of the right side is similar. □

Now we try to get more information about $\max_\theta f_A(\theta)$.

Theorem 2.3. *Let A be a complex matrix of order n with m defined as in Theorem* 2.1. *If* $\operatorname{tr} A^2 = \frac{\operatorname{tr}^2 A}{m}$, *then*

$$\rho(A) \leqslant \frac{|\operatorname{tr} A|}{m} + \left(\frac{m-1}{2m}\left(c(A) - \frac{|\operatorname{tr} A|^2}{m}\right)\right)^{1/2}. \tag{19}$$

Otherwise,

$$\rho(A) \leqslant \max_{-1\leqslant x\leqslant 1}\left\{p_1\sqrt{\frac{1+x}{2}} + p_2\sqrt{\frac{1-x}{2}} + \sqrt{p_3+p_4x}\right\}, \tag{20}$$

where $\beta = \frac{1}{2}\arg\left(\operatorname{tr} A^2 - \frac{\operatorname{tr}^2 A}{m}\right)$, *and*

$$p_1 = \frac{|\Re(\mathrm{e}^{-\mathrm{i}\beta}\operatorname{tr} A)|}{m},\qquad p_2 = \frac{|\Im(\mathrm{e}^{-\mathrm{i}\beta}\operatorname{tr} A)|}{m},$$

$$p_3 = \frac{m-1}{2m}\left(c(A) - \frac{|\operatorname{tr} A|^2}{m}\right),\qquad p_4 = \frac{m-1}{2m}\left|\operatorname{tr} A^2 - \frac{\operatorname{tr}^2 A}{m}\right|.$$

Proof. If $\operatorname{tr} A^2 - \frac{\operatorname{tr}^2 A}{m} = 0$, the inequality (13) yields (19). Otherwise, due to the fact that trigonometric functions are periodic and symmetric, we have

$$\begin{aligned}
\rho(A) &\leqslant \max_\theta f_A(\theta) = \max_\theta f_A(\theta-\beta)\\
&= \max_\theta\left\{\frac{\cos\theta}{m}\Re(\mathrm{e}^{-\mathrm{i}\beta}\operatorname{tr} A) - \frac{\sin\theta}{m}\Im(\mathrm{e}^{-\mathrm{i}\beta}\operatorname{tr} A)\right.\\
&\quad\left. + \sqrt{\frac{m-1}{2m}}\left(c(A) - \frac{|\operatorname{tr} A|^2}{m} + \cos 2\theta\left|\operatorname{tr} A^2 - \frac{\operatorname{tr}^2 A}{m}\right|\right)^{1/2}\right\}\\
&= \max_{0\leqslant\theta\leqslant\frac{\pi}{2}}\left\{p_1\cos\theta + p_2\sin\theta + \sqrt{p_3+p_4\cos 2\theta}\right\}.
\end{aligned}$$

Let $x = \cos 2\theta$. Then (20) is true. □

Theorem 2.4. *Let A be a complex matrix of order n with m defined as in Theorem* 2.1. *If* $\operatorname{tr} A = 0$, *then*

$$\rho(A) \leqslant \sqrt{\frac{m-1}{2m}}\left(\left(\|A\|^4 - \frac{1}{2}\|A^*A - AA^*\|^2\right)^{1/2} + |\operatorname{tr} A^2|\right)^{1/2}. \tag{21}$$

Otherwise,

$$\rho(A) \leqslant \max_{0\leqslant\theta\leqslant\frac{\pi}{2}} \left\{ \frac{\cos\theta}{m}|\mathrm{tr}\, A| + \left(\frac{m-1}{2m}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} \right.\right.$$
$$\left.\left. + |\mathrm{tr}\, A|^2 \left(\cos 2\theta \left(\Re\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right) + \sin 2\theta \left| \Im \frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} \right| \right) \right)^{1/2} \right\}. \quad (22)$$

Proof. If $\mathrm{tr}\, A = 0$, the inequality (21) follows from (13). Otherwise, let $\beta = \arg \mathrm{tr}\, A$, then $\frac{|\mathrm{tr}\, A|^2}{\mathrm{tr}^2 A} = e^{-2i\beta}$. Since trigonometric functions are periodic and symmetric, we have

$$\begin{aligned}
\rho(A) &\leqslant \max_\theta f_A(\theta) = \max_\theta f_A(\theta - \beta) \\
&= \max_\theta \left\{ \frac{\cos\theta}{m}|\mathrm{tr}\, A| + \left(\frac{m-1}{2m}\right)^{1/2} \right. \\
&\quad \left. \times \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + |\mathrm{tr}\, A|^2 \left(\cos 2\theta \left(\Re\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right) - \sin 2\theta \Im \frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} \right) \right)^{1/2} \right\} \\
&= \max_{0\leqslant\theta\leqslant\frac{\pi}{2}} \left\{ \frac{\cos\theta}{m}|\mathrm{tr}\, A| + \left(\frac{m-1}{2m}\right)^{1/2} \right. \\
&\quad \left. \times \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + |\mathrm{tr}\, A|^2 \left(\cos 2\theta \left(\Re\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right) + \sin 2\theta \left| \Im \frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} \right| \right) \right)^{1/2} \right\}.
\end{aligned}$$

□

Corollary 2.2. *Let A be a complex matrix of order n with m defined as in Theorem* 2.1. *Let* $\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} < \frac{1}{m} \leqslant \frac{1}{2}$. *If*

$$c(A) - \frac{|\mathrm{tr}\, A|^2}{m} \leqslant 2m(m-1)|\mathrm{tr}\, A|^2 \left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right)\left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{2m-1}{2m(m-1)} \right), \quad (23)$$

then

$$\rho(A) \leqslant \left(\frac{1}{m-1} - \frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} \right)\sqrt{\frac{m-1}{2m}} \left(\frac{c(A) - \frac{|\mathrm{tr}\, A|^2}{m} - |\mathrm{tr}\, A|^2\left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right)}{\left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right)\left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m-1} \right)} \right)^{1/2}. \quad (24)$$

Otherwise,

$$\rho(A) \leqslant \frac{|\mathrm{tr}\, A|}{m} + \sqrt{\frac{m-1}{2m}} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + |\mathrm{tr}\, A|^2 \left(\frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A} - \frac{1}{m} \right) \right)^{1/2}. \quad (25)$$

Proof. For simplicity, let

$$t = \frac{\mathrm{tr}\, A^2}{\mathrm{tr}^2 A}, \quad d = c(A) - \frac{|\mathrm{tr} A|^2}{m}, \quad s = |\mathrm{tr}\, A|, \quad x = \cos\theta.$$

Define

$$g(x) = \frac{sx}{m} + \sqrt{\frac{m-1}{2m}}(d + s^2(t - 1/m)(2x^2 - 1))^{1/2}.$$

Then its derivative function is

$$g'(x) = \frac{s}{m} + \sqrt{\frac{m-1}{2m}} \frac{2xs^2(t-1/m)}{\left(d + s^2(t-1/m)(2x^2-1)\right)^{1/2}}.$$

Given the conditions available, the inequality (22) is equivalent to

$$\rho(A) \leqslant \max_{0 \leqslant x \leqslant 1} g(x).$$

Notice that $g'(0) \geqslant 0$. By resolving $g'(x) = 0$, we find the unique nonnegative critical point of $g(x)$:

$$x_0 = \left(\frac{d - s^2(t-1/m)}{2s^2 m(m-1)(t-1/m)(t-1/(m-1))} \right)^{1/2}.$$

If $x_0 \leqslant 1$, i.e. (23), $g(x)$ achieves its local maximum $g(x_0)$, which is the right hand of (24). If $x_0 > 1$, $g(x)$ achieves its local maximum $g(1)$, which is the right hand of (25). □

Some results of [1–4] can also be improved by applying Lemma 2.1.

Observing $\sum_{j=1}^{m} |\lambda_j| \leqslant \sqrt{m \sum_{j=1}^{m} |\lambda_j|^2}$, we apply the inequality (9) respectively to $|\lambda_j|$, $\Re\lambda_j$, $\Im\lambda_j$ $(j = 1, 2, \ldots, m)$ and simultaneously use Lemma 2.1. Then we get the following theorem.

Theorem 2.5. *Let A be a complex matrix of order n with λ_j, m defined as in Theorem* 2.1. *Let u_j, v_j respectively denote the real and imaginary parts of the eigenvalues as follows*:

$$u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n \quad \textit{and} \quad v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n.$$

Then, for any $1 \leqslant k \leqslant l \leqslant m$,

$$\frac{1}{l-k+1} \sum_{j=k}^{l} |\lambda_j| \leqslant \left(\frac{c(A)}{m}\right)^{1/2} + \left(\frac{m-l}{ml}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m}\right)^{1/2},$$

$$\frac{1}{l-k+1} \sum_{j=k}^{l} u_j \leqslant \frac{\Re \mathrm{tr}\, A}{m} + \left(\frac{m-l}{2ml}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \Re\left(\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right)\right)^{1/2},$$

$$\frac{1}{l-k+1} \sum_{j=k}^{l} v_j \leqslant \frac{\Im \mathrm{tr}\, A}{m} + \left(\frac{m-l}{2ml}\right)^{1/2} \left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} - \Re\left(\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right)\right)^{1/2}.$$

According to Lemma 2.2 and (16), Theorem 2.5 improves [2, Theorem 3.1, the right side of (3.1)].

As a matter of fact, the proof of [2, Theorem 3.2] also holds for a matrix whose characteristic polynomial is real. Thus, substituting $(c(A) - \Re\mathrm{tr}\, A^2)/2$ for K_C as an upper bound of $\sum(\Im\lambda_j)^2$ in that proof, we extend [2, Theorem 3.2, (3.6)] as below.

Theorem 2.6. *Let A be a complex matrix of ordern with v_j defined as in Theorem* 2.5, *and let the characteristic polynomial of A be real. Let*

$$p = \begin{cases} \left[\frac{n-1}{2}\right] & \textit{if } A \textit{ is nonnegative}, \\ \left[\frac{n}{2}\right] & \textit{otherwise}. \end{cases} \tag{26}$$

Then for $1 \leqslant k \leqslant l \leqslant p$,

$$\frac{1}{l-k+1}\sum_{j=k}^{l} v_j \leqslant \frac{1}{2\sqrt{l}}(c(A) - \Re\mathrm{tr}\, A^2)^{1/2}. \tag{27}$$

Particularly, for any $1 \leqslant j \leqslant n$,

$$|\Im\lambda_j| \leqslant \frac{1}{2}(c(A) - \Re\mathrm{tr}\, A^2)^{1/2}. \tag{28}$$

Applying the inequality (10) respectively to $\Re\lambda_j, \Im\lambda_j$ $(j = 1, 2, \ldots, m)$, and using (5a) and (5b), we obtain

Theorem 2.7. *Let* A *be a complex matrix with* m *defined as in Theorem* 2.1. *Then for any* $1 \leqslant k \leqslant m$,

$$\left|\Re\lambda_k - \frac{\Re\mathrm{tr}\, A}{m}\right| \leqslant \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} + \Re\left(\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right)\right)^{1/2}, \tag{29a}$$

$$\left|\Im\lambda_k - \frac{\Im\mathrm{tr}\, A}{m}\right| \leqslant \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) - \frac{|\mathrm{tr}\, A|^2}{m} - \Re\left(\mathrm{tr}\, A^2 - \frac{\mathrm{tr}^2 A}{m}\right)\right)^{1/2}. \tag{29b}$$

Theorem 2.7, when $m = n$, improves [3, Theorem 6, Theorem 7] by Lemma 2.2 and (16).

Theorem 2.8. *Let* A *be a complex matrix of order* n *with* λ_j, m *defined as in Theorem* 2.1. *If its characteristic polynomial is real, then* λ_j $(j = 1, 2, \ldots, m)$ *lie in the elliptic region*

$$\frac{\left(x - \frac{\mathrm{tr}\, A}{m}\right)^2}{r_x^2} + \frac{y^2}{r_y^2} \leqslant 1, \tag{30}$$

where

$$r_x = \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) + \mathrm{tr} A^2 - \frac{2\mathrm{tr}^2 A}{m}\right)^{1/2},$$

$$r_y = \left(\frac{m-1}{2m}\right)^{1/2}\left(c(A) - \mathrm{tr}\, A^2\right)^{1/2}.$$

Proof. By [4, Theorem 3.6], the eigenvalues $\lambda_j (1 \leqslant j \leqslant m)$ lie in the elliptic region defined by

$$\frac{\left(x - \frac{\mathrm{tr}\, A}{m}\right)^2}{\rho_x^2} + \frac{y^2}{\rho_y^2} \leqslant 1,$$

where

$$\rho_x = \left(\frac{m-1}{m}\right)^{1/2}\left(\sum_{j=1}^{m}(\Re\lambda_j)^2 - \frac{(\mathrm{tr}\, A)^2}{m}\right)^{1/2},$$

$$\rho_y = \left(\frac{m-1}{m}\sum_{j=1}^{m}(\Im\lambda_j)^2\right)^{1/2}.$$

By (5a) and (5b), we have $\rho_x \leqslant r_x$ and $\rho_y \leqslant r_y$. Therefore, the eigenvalues $\lambda_j (1 \leqslant j \leqslant m)$ are contained in the elliptic region determined by (30). $\square$

Theorem 2.8 improves [4, Corollary 3.7, Corollary 3.8].

3. Lower bounds for the spectral radius

In this section we give some lower bounds for the spectral radius using traces.

Theorem 3.1. *Let A be a complex matrix of order n with* λ_j, m *defined as in Theorem* 2.1. *Then*

$$\rho(A) \geqslant \left(\frac{|\mathrm{tr}^2 A - \mathrm{tr}\, A^2|}{m(m-1)}\right)^{1/2}, \quad m \geqslant 2; \tag{31}$$

$$\rho(A) \geqslant \left(\frac{|2\mathrm{tr}\, A^3 - 3\mathrm{tr}\, A^2 \cdot \mathrm{tr}\, A + \mathrm{tr}^3 A|}{m(m-1)(m-2)}\right)^{1/3}, \quad m \geqslant 3. \tag{32}$$

Proof. Let λ_j $(j = 1, 2, \ldots, n)$ be defined as in Theorem 2.1. Notice that $\lambda_j = 0 (j > m)$. Let

$$s_t = \sum_{j=1}^{m} \lambda_j^t,$$

$$p(x) = \prod_{j=1}^{m} (x - \lambda_j) = x^m - \sigma_1 x^{m-1} + \cdots + (-1)^m \sigma_m.$$

According to Newton's formulas [8, p200], we have

$$\sigma_2 = \frac{s_1^2 - s_2}{2}, \quad m \geqslant 2;$$

$$\sigma_3 = \frac{s_3 - \sigma_1 s_2 + \sigma_2 s_1}{3} = \frac{2s_3 - 3s_2 s_1 + s_1^3}{6}, \quad m \geqslant 3.$$

Observing that $s_t = \mathrm{tr}\, A^t$ and $\sigma_t = \sum \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_t}$ (sum of all products of possible combination of distinct t terms of roots of $p(x)$) due to Vieta's Theorem, we see

$$|\sigma_2| = |\mathrm{tr}^2 A - \mathrm{tr}\, A^2|/2 \leqslant |\lambda_1^2| m(m-1)/2 = \rho^2(A) m(m-1)/2,$$

$$\begin{aligned} |\sigma_3| &= |2\mathrm{tr}\, A^3 - 3\mathrm{tr}\, A^2 \cdot \mathrm{tr}\, A + \mathrm{tr}^3 A|/6 \\ &\leqslant |\lambda_1^3| m(m-1)(m-2)/6 \\ &= \rho^3(A) m(m-1)(m-2)/6. \end{aligned}$$

Therefore, the inequalities (31) and (32) hold. $\square$

Theorem 3.2. *Let A be a complex matrix of order n with* λ_j, m *defined as in Theorem* 2.1. *If A is normal*, *then*

$$\rho(A) \geqslant \frac{\|A\|}{\sqrt{m}}. \tag{33}$$

If A is skew-symmetric, then

$$\rho(A) \geqslant \begin{cases} \frac{\|A\|}{\sqrt{n}}, & \text{if } n \text{ is even}, \\ \frac{\|A\|}{\sqrt{n-1}}, & \text{if } n \text{ is odd}. \end{cases} \tag{34}$$

Proof. Since

$$\sum_{j=1}^{m} |\lambda_j|^2 \leqslant m|\lambda_1|^2 = m\rho^2(A),$$

and $\sum_{j=1}^{m} |\lambda_j|^2 = \|A\|^2$ due to normality of A, inequality (33) holds. The inequality (34) follows from the fact that the rank of a skew-symmetric matrix can only be even. □

Theorem 3.3. *Let A be a complex matrix of order n with λ_j, m defined as in Theorem 2.1. Then*

$$\rho(A) \geqslant \left(\frac{|\mathrm{tr}\, A \cdot \mathrm{tr}\, A^2 - \mathrm{tr}\, A^3|}{m(m-1)} \right)^{1/3}, \quad m \geqslant 2. \tag{35}$$

Proof. Since

$$\begin{aligned} |\mathrm{tr}\, A \cdot \mathrm{tr}\, A^2 - \mathrm{tr}\, A^3| &= \left| \left(\sum_{k=1}^{m} \lambda_k\right)\left(\sum_{l=1}^{m} \lambda_l^2\right) - \sum_{j=1}^{m} \lambda_j^3 \right| \\ &= \left| \sum_{k \neq l} \lambda_k \lambda_l^2 \right| \\ &\leqslant m(m-1)|\lambda_1^3| = m(m-1)\rho^3(A), \end{aligned}$$

the inequality (35) holds. □

The inequality (31) extends [9, Theorem 1, (4)], and the inequality (34) improves [9, Corollary 2, Corollary 3].

4. Examples

Example 1 [2, *Example* 4.1]. Wolkowicz and Styan estimate eigenvalues for the matrix

$$A = \begin{pmatrix} 7+3\mathrm{i} & -4-6\mathrm{i} & -4 \\ -1-6\mathrm{i} & 7 & -2-6\mathrm{i} \\ 2 & 4-6\mathrm{i} & 13-3\mathrm{i} \end{pmatrix}.$$

Define λ_j, u_j, v_j as in Theorem 2.5. Their bounds include:

$$\begin{aligned} &|\lambda_1| \leqslant 24.91, \quad |\lambda_2| \leqslant 18.73, \quad |\lambda_3| \leqslant 11.03, \\ &(|\lambda_1| + |\lambda_2| + |\lambda_3|)/3 \leqslant 12.55, \\ &u_1 \leqslant 13.81, \quad u_2 \leqslant 11.40, \quad u_3 \leqslant 9, \\ &v_1 \leqslant 11.49, \quad v_2 \leqslant 5.74, \quad v_3 \leqslant -3.83, \end{aligned}$$

while applying Theorem 2.5 with $m = 3$, we obtain

$$|\lambda_1| \leqslant 24.2994, \quad |\lambda_2| \leqslant 18.3229, \quad |\lambda_3| \leqslant 12.3465,$$
$$(|\lambda_1| + |\lambda_2| + |\lambda_3|)/3 \leqslant 12.3465,$$
$$u_1 \leqslant 13.1756, \quad u_2 \leqslant 11.0878, \quad u_3 \leqslant 9,$$
$$v_1 \leqslant 11.1998, \quad v_2 \leqslant 5.5999, \quad v_3 \leqslant 0.$$

The upper bound for the spectral radius by [1, Theorem 3.1, (3.12)] is 21.7279. The inequality (15) gives a tighter estimate 20.1998. Moreover, the bounds for the spectral radius by Theorem 2.3 and Corollary 2.2 are both 14.8159. Their coincidence is necessary because both of them are the maximum of the same function. The spectral radius is actually 12.7279.

Example 2 [3, *Example* 4.1]. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

From [3, Theorem 6], the eigenvalues λ_j satisfy

$$|\lambda_j - 1| \leqslant 1.77169,$$
$$|\Re\lambda_j - 1| \leqslant 1.25277,$$
$$|\Im\lambda_j| \leqslant 1.25277.$$

By inequalities (12), (29a) and (29a) with $m = 3$, we obtain

$$|\lambda_j - 1| \leqslant 1.18921.$$
$$|\Re\lambda_j - 1| \leqslant 1.18921, \quad |\Im\lambda_j| \leqslant 1.18921.$$

Example 3 [4, *Example* 4.3]. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The elliptic region containing eigenvalues of A by [4, Corollary 2.4] is

$$\frac{(x - 0.25)^2}{1.489488^2} + \frac{y^2}{1.776534^2} \leqslant 1.$$

From Theorem 2.8 with $m = 4$, we get a smaller region to cover the spectrum of A:

$$\frac{(x - 0.25)^2}{1.478025^2} + \frac{y^2}{1.766934^2} \leqslant 1.$$

[4, Corollary 2.4] presents the following elliptic region covering nonzero eigenvalues:

$$\frac{(x - 0.5)^2}{1.163636^2} + \frac{y^2}{1.450534^2} \leqslant 1.$$

Our corresponding region by Theorem 2.8 with $m = 2$ is

$$\frac{(x - 0.5)^2}{1.153851^2} + \frac{y^2}{1.442696^2} \leqslant 1.$$

Example 4. Though our lower bounds have no absolute advantage, they can sometimes perform well. Let

$$A = \begin{pmatrix} -4 & -4 & -8 & -4 \\ -2 & -7 & 2 & 3 \\ 0 & -1 & 3 & 4 \\ 3 & -1 & -4 & 7 \end{pmatrix}.$$

We calculate lower bounds of $\rho(A)$: 0.2500 by [2, Theorem 3.1, (3.2a)], 2.7122 by [9, Theorem 1, Theorem 3]. Let $m = 4$ here. Then our lower bounds are

$$\begin{aligned} &2.4495 \quad \text{by (31)}, \\ &4.6570 \quad \text{by (32)}, \\ &4.7027 \quad \text{by (35)}. \end{aligned}$$

Actually, the true spectral radius is 8.8934.

Acknowledgments

The authors would like to express their thankfulness to the referees and editor for their detailed and helpful suggestions for revising the manuscript.

References

[1] Henry Wolkowicz, George P.H. Styan, Bounds for eigenvalues using traces, Linear Algebra Appl. 29 (1980) 471–506.
[2] Henry Wolkowicz, George P.H. Styan, More bounds for eigenvalues using traces, Linear Algebra Appl. 31 (1980) 1–17.
[3] Oscar Rojo, Ricardo L. Soto, Hector Rojo, New eigenvalue estimates for complex matrices, Comput. Math. Appl. 25 (3) (1993) 91–97.
[4] Oscar Rojo, Ricardo L. Soto, T. Avila, Hector Rojo, Localization of eigenvalues in elliptic regions, Comput. Math. Appl. 29 (7) (1995) 3–11.
[5] P.J. Eberlein, On measures of non-normality for matrices, Amer. Math. Month. 72 (1965) 995–996.
[6] R. Kress, H.L. de Vries, R. Wegmann, On nonnormal matrices, Linear Algebra Appl. 8 (1974) 109–120.
[7] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[8] M.S. Yao, Advanced Algebra, Fudan University Press, 2003.
[9] Bill G. Horne, Lower bounds for the spectral radius of a matrix, Linear Algebra Appl. 263 (1997) 261–273.