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We derive an inclusion region for the eigenvalues of a matrix that can be considered an alternative to the Brauer set. It is accompanied by non-singularity conditions.

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AMS Subject Classification: 15A18

1. Introduction

A well-known way to obtain inclusion regions for eigenvalues of matrices is the use of Geršgorin disks. It is stated in the following theorem, for which we first define the deleted absolute row and column sums $R_i'(A)$ and $C_i'(A)$, respectively, of a matrix $A \in \mathbb{C}^{n\times n}$ with elements a_{ij} as:

$$R_i'(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad \text{and} \quad C_i'(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}|.$$

The dependence on the matrix A will be left out of our notation whenever there can be no confusion about what matrix is being considered. The theorem is then given as follows:

Theorem 1.1 [6] *All the eigenvalues of the $n \times n$ complex matrix A are located in the union of the n disks*

$$\Gamma^R = \bigcup_{i=1}^{n} \Gamma_i^R,$$

where

$$\Gamma_i^R = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq R_i' \right\}$$

and also in the union of the n disks

$$\Gamma^C = \bigcup_{j=1}^{n} \Gamma_j^C,$$

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where

$$\Gamma_j^C = \left\{ z \in \mathbb{C} \,:\, |z - a_{jj}| \le C'_j \right\}.$$

The theorem has a row and column version because the spectra of A and A^T are identical. We refer to, for example [7, Chap. 6] for related results.

In addition to the Geršgorin set, there exists another well-known inclusion set for the eigenvalues, namely the Brauer set, as stated in the following theorem:

THEOREM 1.2 [1] *All the eigenvalues of the $n \times n$ complex matrix A are located in the union of the $\binom{n}{2}$ ovals of Cassini*

$$\Delta^R = \bigcup_{\substack{i,j=1 \\ i \ne j}}^{n} \left\{ z \in \mathbb{C} \,:\, |z - a_{ii}||z - a_{jj}| \le R'_i R'_j \right\},$$

and also in the union of the $\binom{n}{2}$ ovals of Cassini

$$\Delta^C = \bigcup_{\substack{i,j=1 \\ i \ne j}}^{n} \left\{ z \in \mathbb{C} \,:\, |z - a_{ii}||z - a_{jj}| \le C'_i C'_j \right\}.$$

It is an easily established and well-known fact that $\Delta^R \subseteq \Gamma^R$ and $\Delta^C \subseteq \Gamma^C$.

More complicated sets can be derived (see, e.g. [2–5,8–11], and references therein), although some of these involve the union of a very large number of sets.

As is the case for the Geršgorin and Brauer sets, our results can be improved by using a suitable similar matrix $S^{-1}AS$ instead of A, which has the same eigenvalues (see, e.g. [7, Chap. 6]).

We need a few more definitions, in addition to the definition of R'_i and C'_i. They all relate to a matrix $A \in \mathbb{C}^{n \times n}$ and they are listed below:

$$R'_i = \sum_{\substack{k=1 \\ k \ne i}}^{n} |a_{ik}| = \sum_{k=1}^{n} |a_{ik}| - |a_{ii}|$$

$$R'_{ij} = \sum_{\substack{k=1 \\ k \ne j}}^{n} |a_{ik}| = \sum_{k=1}^{n} |a_{ik}| - |a_{ij}|$$

$$R''_{ij} = \sum_{\substack{k=1 \\ k \ne i,j}}^{n} |a_{ik}| = \sum_{k=1}^{n} |a_{ik}| - |a_{ii}| - |a_{ij}| = R'_i - |a_{ij}|$$

$$C'_i = \sum_{\substack{k=1 \\ k \ne i}}^{n} |a_{ki}| = \sum_{k=1}^{n} |a_{ki}| - |a_{ii}|$$

$$C'_{ji} = \sum_{\substack{k=1 \\ k \ne j}}^{n} |a_{ki}| = \sum_{k=1}^{n} |a_{ki}| - |a_{ji}|$$

$$C''_{ji} = \sum_{\substack{k=1 \\ k \ne i,j}}^{n} |a_{ki}| = \sum_{k=1}^{n} |a_{ki}| - |a_{ii}| - |a_{ji}| = C'_i - |a_{ji}|.$$

We derive our spectral inclusion region in Section 2, along with accompanying non-singularity conditions. Some examples are presented in Section 3.

2. An alternative to the Brauer set

The next theorem contains our main result, namely a spectral inclusion set which, like the Brauer set, is composed of $\binom{n}{2}$ oval-like sets.

THEOREM 2.1 *All the eigenvalues of the $n \times n$ complex matrix A are located in the union of the following $\binom{n}{2}$ sets*:

$$\Omega^R = \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left(\Omega^R_{ij} \cap \Omega^R_{ji}\right),$$

where

$$\Omega^R_{ij} = \left\{ z \in \mathbb{C} : \left|(z-a_{ii})(z-a_{jj}) - a_{ij}a_{ji}\right| \le |z-a_{jj}|R''_{ij} + |a_{ij}|R''_{jt} \right\},$$

and also in the union of the following $\binom{n}{2}$ sets:

$$\Omega^C = \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left(\Omega^C_{ij} \cap \Omega^C_{ji}\right),$$

where

$$\Omega^C_{ij} = \left\{ z \in \mathbb{C} : \left|(z-a_{ii})(z-a_{jj}) - a_{ij}a_{ji}\right| \le |z-a_{jj}|C''_{ji} + |a_{ji}|C''_{ij} \right\}.$$

Proof We only prove the row version because the column version's proof is entirely analogous. Let λ be an eigenvalue of A with corresponding eigenvector x, i.e. $Ax = \lambda x$. Since x is an eigenvector, it has at least one non-zero component. Define x_μ as the component of x with the largest absolute value, so that $|x_\mu| \ge |x_i|$ for all $i = 1, 2, \ldots, n$ and $x_\mu \neq 0$.

If x_μ is the only non-zero component of x, then we have

$$A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_\mu \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_\mu \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which implies that $a_{i\mu} = 0$ for all $i \neq \mu$, and $\lambda = a_{\mu\mu}$. In this case, and for $i \neq \mu$, we have that

$$\Omega^R_{i\mu} = \left\{ z \in \mathbb{C} : \left|(z-a_{ii})(z-a_{\mu\mu})\right| \le |z-a_{\mu\mu}|R''_{i\mu} \right\},$$

and

$$\Omega^R_{\mu i} = \left\{ z \in \mathbb{C} : \left|(z - a_{ii})(z - a_{\mu\mu})\right| \leq |z - a_{ii}|R''_{\mu i} + |a_{\mu i}|R''_{i\mu} \right\},$$

so that $\lambda = a_{\mu\mu} \in \Omega^R_{i\mu} \cap \Omega^R_{\mu i} \subseteq \Omega^R$.

Assume now that there is at least one other non-zero component of x. Define x_ν as the component of x with the second largest absolute value, i.e. $|x_\mu| \geq |x_\nu| \geq |x_i|$ for all $i = 1, 2, \ldots, n$, $i \neq \mu$, $\mu \neq \nu$, and x_μ, $x_\nu \neq 0$. We then have

$$\lambda x_\mu = \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\mu j} x_j + a_{\mu\mu} x_\mu + a_{\mu\nu} x_\nu$$

$$\lambda x_\nu = \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\nu j} x_j + a_{\nu\mu} x_\mu + a_{\nu\nu} x_\nu,$$

which is equivalent to

$$(\lambda - a_{\mu\mu}) x_\mu - a_{\mu\nu} x_\nu = \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\mu j} x_j$$

$$-a_{\nu\mu} x_\mu + (\lambda - a_{\nu\nu}) x_\nu = \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\nu j} x_j.$$

Solving for x_μ and x_ν, we obtain

$$((\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu} a_{\nu\mu}) x_\mu = (\lambda - a_{\nu\nu}) \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\mu j} x_j + a_{\mu\nu} \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\nu j} x_j \tag{1}$$

$$((\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu} a_{\nu\mu}) x_\nu = (\lambda - a_{\mu\mu}) \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\nu j} x_j + a_{\nu\mu} \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} a_{\mu j} x_j. \tag{2}$$

Taking absolute values of (1) and (2) and using the triangle inequality yields

$$\left|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu} a_{\nu\mu}\right| |x_\mu| \leq |\lambda - a_{\nu\nu}| \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} |a_{\mu j}||x_j| + |a_{\mu\nu}| \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} |a_{\nu j}||x_j|$$

$$|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu} a_{\nu\mu}||x_\nu| \leq |\lambda - a_{\mu\mu}| \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} |a_{\nu j}||x_j| + |a_{\nu\mu}| \sum_{\substack{j=1 \\ j \neq \mu,\nu}}^{n} |a_{\mu j}||x_j|.$$

Since $x_\mu \neq 0$ and $x_\nu \neq 0$ are, in absolute value, the largest and second largest components of x, respectively, and since these components do not appear in the right-hand side of these inequalities, we can divide through by their absolute values to obtain

$$|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu} a_{\nu\mu}| \leq |\lambda - a_{\nu\nu}| R''_{\mu\nu} + |a_{\mu\nu}| R''_{\nu\mu}$$

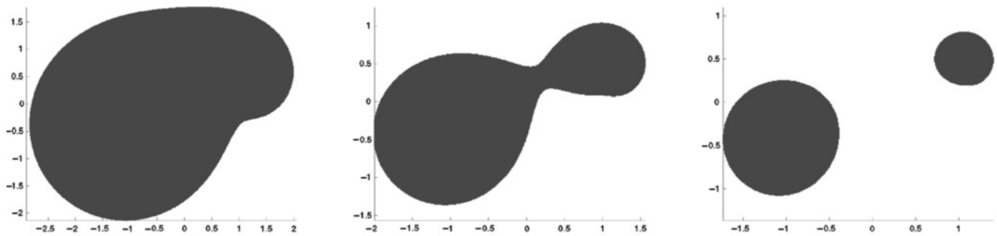

Figure 1. Examples of the Ω_{ij} sets.

and

$$|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu}a_{\nu\mu}| \leq |\lambda - a_{\mu\mu}|R''_{\nu\mu} + |a_{\nu\mu}|R''_{\mu\nu}.$$

The eigenvalue λ satisfies both these inequalities, but we do not know which μ and ν correspond to a given eigenvalue. We can, therefore, only say that any eigenvalue must lie in the union of all possible sets described by the above inequalities. ∎

The sets Ω_{ij}^R and Ω_{ij}^C have properties similar to the Cassinian ovals of the Brauer sets, as can be seen in a few typical examples of these sets in Figure 1.

In the next theorem, we show that the Ω_{ij} sets are contained in Geršgorin disks.

Theorem 2.2 *The sets Ω_{ij}^R and Ω_{ij}^C satisfy, for all $i \neq j$, that*

$$\Omega_{ij}^R \subseteq \Gamma_i^R \cup \Gamma_j^R \quad \textit{and} \quad \Omega_{ij}^C \subseteq \Gamma_i^C \cup \Gamma_j^C,$$

so that the sets Ω^R and Ω^C satisfy

$$\Omega^R \subseteq \Gamma^R \quad \textit{and} \quad \Omega^C \subseteq \Gamma^C.$$

Proof We prove the row version of the theorem. The column version follows analogously.

Assume that $z \in \Omega_{ij}^R$ for some $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Then $z \in \Gamma_i^R$ or $z \notin \Gamma_i^R$. If $z \in \Gamma_i^R$, there is nothing to prove. If $z \notin \Gamma_i^R$, then

$$|z - a_{jj}|R''_{ij} + |a_{ij}|R''_{ji} \geq \big|(z - a_{ii})(z - a_{jj}) - a_{ij}a_{ji}\big| \geq |z - a_{ii}||z - a_{jj}| - |a_{ij}||a_{ji}|. \tag{3}$$

Since $R''_{ij} = R'_i - |a_{ij}|$ and $R''_{ij} = R'_j - |a_{ji}|$, and because $|z - a_{ii}| > R'_i$, we have from (3) that

$$|z - a_{jj}|(R'_i - |a_{ij}|) + |a_{ij}|(R'_j - |a_{ji}|) \geq R'_i|z - a_{jj}| - |a_{ij}||a_{ji}|,$$

which implies that

$$|a_{ij}|R'_j \geq |a_{ij}||z - a_{jj}|.$$

If $a_{ij} \neq 0$, then this inequality means that $z \in \Gamma_j^R$. In other words, if z is not in Γ_i^R, then it must be in Γ_j^R. If $a_{ij} = 0$, then it is easy to see that $\Omega_{ij}^R = \Gamma_i^R \cup \{a_{jj}\} \subseteq \Gamma_i^R \cup \Gamma_j^R$. As an immediate consequence of the above, we have that $\Omega^R \subseteq \Gamma^R$. This completes the proof. ∎

The following theorem shows that not only do our new sets lie in the Geršgorin sets, but they are contained in the Brauer sets as well.

Theorem 2.3 *The sets Ω_{ij}^R and Ω_{ij}^C satisfy for all $i \neq j$ that*

$$\Omega_{ij}^R \cap \Omega_{ji}^R \subseteq \Delta_{ij}^R \quad \text{and} \quad \Omega_{ij}^C \cap \Omega_{ji}^C \subseteq \Delta_{ij}^C,$$

so that $\Omega^R \subseteq \Delta^R$ and $\Omega^C \subseteq \Delta^C$.

Proof Once again, we prove the row version of the theorem while the column version follows analogously. Pick any $z \in \Omega_{ij}^R \cap \Omega_{ji}^R$; then we know from Theorem 2.2 that $z \in \Gamma_i^R$ or that $z \in \Gamma_j^R$. Assume that $z \in \Gamma_j^R$. Because $z \in \Omega_{ij}^R$, we have that

$$\left|(z-a_{ii})(z-a_{jj}) - a_{ij}a_{ji}\right| \leq |z-a_{jj}|R_{ij}'' + |a_{ij}|R_{ji}''.$$

Since

$$|z-a_{ii}||z-a_{jj}| - |a_{ij}||a_{ji}| \leq |(z-a_{ii})(z-a_{jj}) - a_{ij}a_{ji}|,$$

z must satisfy

$$|z-a_{ii}||z-a_{jj}| - |a_{ij}||a_{ji}| \leq |z-a_{jj}|R_{ij}'' + |a_{ij}|R_{ji}'',$$

or

$$|z-a_{ii}||z-a_{jj}| \leq |z-a_{jj}|R_{ij}'' + |a_{ij}|\left(R_{ji}'' + |a_{ji}|\right). \tag{4}$$

Because $z \in \Gamma_j^R$, it satisfies $|z-a_{jj}| \leq R_j'$, and since $R_{ji}'' = R_j' - |a_{ji}|$ and $R_{ji}'' = R_i' - |a_{ij}|$, we obtain from (4) that

$$|z-a_{ii}||z-a_{jj}| \leq R_j'R_{ij}'' + |a_{ij}|\left(R_{ji}'' + |a_{ji}|\right) = R_j'R_{ij}'' + |a_{ij}|R_j' = R_i'R_j'.$$

This means that $z \in \Delta_{ij}^R$. If z lies in Γ_i^R instead of Γ_j^R, then we repeat the above argument with Ω_{ji}^R instead of Ω_{ij}^R to reach the same conclusion, i.e. we have shown that $\Omega_{ij}^R \cap \Omega_{ji}^R \subseteq \Delta_{ij}^R$.

As an immediate consequence, we obtain that $\Omega^R \subseteq \Delta^R$. This concludes the proof. ■

Remarks

(1) The proof of Theorem 2.3 relies on the first part of Theorem 2.2 but, once Theorem 2.3 is proved, the second part of Theorem 2.2, namely $\Omega^R \subseteq \Gamma^R$ and $\Omega^C \subseteq \Gamma^C$, also follows because $\Delta^R \subseteq \Gamma^R$ and $\Delta^C \subseteq \Gamma^C$.

(2) Whereas the diagonal elements of the matrix are always contained in the Brauer sets, the same is not true for the new sets: consider, for example, a_{ii}, which trivially lies in Δ_{ij}^R. For a_{ii} to be contained in $\Omega_{ij}^R \cap \Omega_{ji}^R$, we must have

$$|a_{ij}||a_{ji}| \leq |a_{ii} - a_{jj}|R_{ij}'' + |a_{ij}|R_{ji}'' \qquad \text{and} \qquad |a_{ij}||a_{ji}| \leq |a_{ji}|R_{ij}'',$$

but these inequalities are not necessarily satisfied. To take a simple case, assume that a_{ij}, $a_{ji} \neq 0$ and that $a_{ii} = a_{jj}$. The resulting inequalities become

$$|a_{ji}| \leq R_{ji}'' \qquad \text{and} \qquad |a_{ij}| \leq R_{ij}'',$$

which are clearly not satisfied in general. On the other hand, $a_{ij} = 0$ or $a_{ji} = 0$ is obviously a sufficient condition for both a_{ii} and a_{jj} to be contained in $\Omega_{ij}^R \cap \Omega_{ji}^R$.

An eigenvalue inclusion set for a matrix naturally leads to non-singularity conditions for that matrix by requiring that $z = 0$ not be included in the set. The following two theorems present such conditions, the second one requiring slightly more work than the first.

THEOREM 2.4 *Let $A \in \mathbb{C}^{n\times n}$, then A is invertible if $\forall i, j = 1, 2, \ldots, n$ and $i \neq j$:*

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| > \min\left\{|a_{jj}|R''_{ij} + |a_{ij}|R''_{ji}, |a_{ii}|R''_{ji} + |a_{ji}|R''_{ij}\right\}.$$

Proof A sufficient condition for the invertibility of the matrix A is $0 \notin \Omega^R$. By Theorem 2.1, this condition will be fulfilled if for any $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$:

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| > |a_{jj}|R''_{ij} + |a_{ij}|R''_{ji}$$

or

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| > |a_{ii}|R''_{ji} + |a_{ji}|R''_{ij}.$$

Since the left-hand side is the same in both these inequalities, the proof follows. ∎

THEOREM 2.5 *Let $A \in \mathbb{C}^{n\times n}$, then A is invertible if $\forall i, j = 1, 2, \ldots, n$ and $i \neq j$:*

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| > \min\left\{\sum_{\substack{k=1\\k\neq i,j}}^{n}\left|a_{jj}a_{ik} - a_{ij}a_{jk}\right|, \sum_{\substack{k=1\\k\neq i,j}}^{n}\left|a_{ii}a_{jk} - a_{ji}a_{ik}\right|\right\}.$$

The proof follows from Equations (1) and (2) in the proof of Theorem 2.1, which we can rewrite as

$$\big((\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu}a_{\nu\mu}\big)x_\mu = \sum_{\substack{k=1\\k\neq\mu,\nu}}^{n}\big((\lambda - a_{\nu\nu})a_{\mu k} + a_{\mu\nu}a_{\nu k}\big)x_k$$

$$\big((\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu}a_{\nu\mu}\big)x_\nu = \sum_{\substack{k=1\\k\neq\mu,\nu}}^{n}\big((\lambda - a_{\mu\mu})a_{\nu k} + a_{\nu\mu}a_{\mu k}\big)x_k.$$

Analogously as in the proof of Theorem 2.1, we then obtain

$$\left|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu}a_{\nu\mu}\right| \leq \sum_{\substack{k=1\\k\neq\mu,\nu}}^{n}\left|(\lambda - a_{\nu\nu})a_{\mu k} + a_{\mu\nu}a_{\nu k}\right| \tag{5}$$

$$\left|(\lambda - a_{\mu\mu})(\lambda - a_{\nu\nu}) - a_{\mu\nu}a_{\nu\mu}\right| \leq \sum_{\substack{k=1\\k\neq\mu,\nu}}^{n}\left|(\lambda - a_{\mu\mu})a_{\nu k} + a_{\nu\mu}a_{\mu k}\right|. \tag{6}$$

If the matrix is to be non-singular, then it is sufficient for $\lambda = 0$ not to be in the intersection of the two regions determined by inequalities (5) and (6) for any pair (μ, ν) with $\mu \neq \nu$. This means that

$$\left|a_{\mu\mu}a_{\nu\nu} - a_{\mu\nu}a_{\nu\mu}\right| > \sum_{\substack{k=1\\k\neq\mu,\nu}}^{n}\left|a_{\nu\nu}a_{\mu k} - a_{\mu\nu}a_{\nu k}\right|$$

or

$$\left|a_{\mu\mu}a_{\nu\nu} - a_{\mu\nu}a_{\nu\mu}\right| > \sum_{\substack{k=1 \\ k\neq\mu,\nu}}^{n}\left|a_{\mu\mu}a_{\nu k} - a_{\nu\mu}a_{\mu k}\right|,$$

for any given pair (μ, ν) with $\mu \neq \nu$. Since the left-hand side is the same in both these inequalities, the proof follows with $\mu = i$ and $\nu = j$. ∎

We note that the triangle inequality for the absolute value implies that the conditions in Theorem 2.5 are weaker than those in Theorem 2.4.

3. Examples

To conclude, we present a few small examples, in which we have graphed Δ^R, the Brauer set, and Ω^R, our alternative to it, for a few matrices. The Brauer sets Δ^R are shaded in light grey, the sets Ω^R are shaded in dark grey, and the eigenvalues appear as white dots (Figures 2 and 3). As one can see, Ω^R can be quite similar to Δ^R, as for A_1, but it can also be significantly smaller, as is the case for the other matrices. The matrix A_4 is taken from [4].

$$A_1 = \begin{pmatrix} 5-4i & -2-3i & 4-2i & 2-i \\ 3-3i & -4i & 2-2i & -1-3i \\ -4i & 4i & 3-i & -4+3i \\ 1+4i & 5+4i & -2-2i & i \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -4-i & -4+3i & 4+5i & 12 \\ 4+3i & 2-i & 4+3i & 2+4i \\ -4i & 1-i & 4i & 3+5i \\ -10 & 1+4i & -2+5i & -3-2i \end{pmatrix}.$$

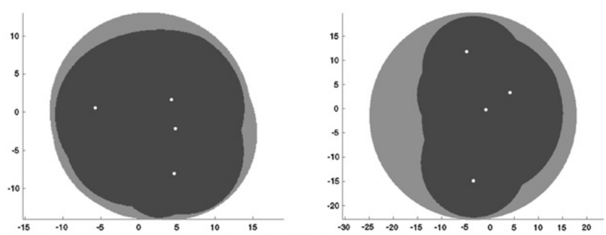

Figure 2. The sets Δ^R and Ω^R for the matrices A_1 (left) and A_2 (right).

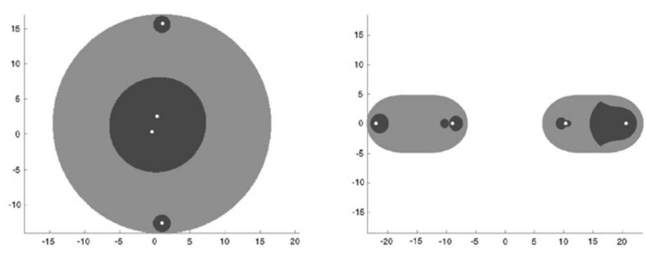

Figure 3. The sets Δ^R and Ω^R for the matrices A_3 (left) and A_4 (right).

$$A_3 = \begin{pmatrix} 1+2i & i & i & 20 \\ 1 & 2i & i & 1-i \\ 1-i & 1+i & i & -i \\ -10 & -i & 0 & 1+i \end{pmatrix}; \quad A_4 = \begin{pmatrix} 10 & 0 & 3 & 5 \\ 0 & -10 & 2 & 4 \\ 1 & 5 & 20 & 0 \\ 4 & 4 & 0 & -20 \end{pmatrix}.$$

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