



SMALLER GERSHGORIN DISKS FOR MULTIPLE EIGENVALUES OF COMPLEX MATRICES

I. BÁRÁNY^{1,2,*,\dagger,\ddagger} and J. SOLYMOSI^{3,4,\dagger,\S}

¹Rényi Institute of Mathematics, 13-15 Reáltanoda Street, Budapest, 1053, Hungary
e-mail: barany@renyi.hu

²Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

³Department of Mathematics, University of British Columbia, 1984 Mathematics Rd, Vancouver, BC V6T 1Z2, Canada
e-mail: solymosi@math.ubc.ca

⁴Óbuda University, 96/B Bécsi Street, Budapest, 1034, Hungary

(Received November 5, 2022; revised December 19, 2022; accepted December 28, 2022)

Abstract. Extending an earlier result for real matrices we show that multiple eigenvalues of a complex matrix lie in a reduced Gershgorin disk. One consequence is a slightly better estimate in the real case. Another one is a geometric application. Further results of a similar type are given for normal and almost symmetric matrices.

1. Introduction

Gershgorin's classic result [9] (see [12] as well) has been a major tool to estimate the eigenvalues of an $n \times n$ (complex) matrix $A = (a_{i,j})_{i,j=1}^n$ for the last 90 years. It says that if λ is an eigenvalue of A , then there is $i \in [n] := \{1, \dots, n\}$ such that

$$|a_{i,i} - \lambda| \leq \sum_{j \neq i} |a_{i,j}|.$$

* Corresponding author.

^{\dagger} Research of both authors was supported by a Hungarian National Research Grant KKP, Grant no. 133819.

^{\ddagger} Research of IB was supported in part by Hungarian National Research grants no. 131529, 131696.

^{\S} Research of JS was supported in part by an NSERC Discovery grant and OTKA K grant no. 119528.

Key words and phrases: matrix, eigenvalue, Gershgorin disk.

Mathematics Subject Classification: 15A18, 15A42.

In other words λ lies in the *Gershgorin disk* $D(a_{i,i}, R_i)$ in the complex plane for some $i \in [n]$, where $a_{i,i}$ is the centre, and $R_i = \sum_{j \neq i} |a_{i,j}|$ is the radius of the disk. Equivalently, every eigenvalue λ of A satisfies

$$(1.1) \quad \lambda \in \bigcup_{i=1}^n D(a_{i,i}, R_i).$$

In other words $\lambda \notin \bigcup_{i=1}^n D(a_{i,i}, R_i)$ implies that $\text{rank}(A - \lambda I) = n$, where I stands for the $n \times n$ unit matrix.

In particular, if

$$(1.2) \quad |a_{i,i}| > \sum_{j \neq i} |a_{i,j}| \quad \text{for every } i \in [n],$$

then $0 \notin \bigcup_{i=1}^n D(a_{i,i}, R_i)$ and so the matrix is not singular. Matrices satisfying (1.2) are called *diagonally dominant* matrices. This is a classical application of Gershgorin's theorem: diagonally dominant matrices have full rank, that is, zero is not an eigenvalue because no Gershgorin disk contains the origin. A good source of information on Gershgorin's theorem is the excellent book by R. Varga [12]. For more recent extensions and generalizations of Gershgorin's theorem see for instance Alon [2] and Alon and Solymosi [3].

Our main result is an extension or strengthening of Gershgorin's theorem for eigenvalues with multiplicity two or more.

THEOREM 1.1. *Let A be an $n \times n$ complex matrix and λ be a complex number. If there are $c_i \in \mathbb{C}$ satisfying the inequality*

$$(1.3) \quad |a_{i,i} - c_i - \lambda| > \sum_{i \neq j} |a_{i,j} - c_i| \quad \text{for all } i \in [n],$$

then $\text{rank}(A - \lambda I) \geq n - 1$.

The relation to Gershgorin disks can be seen as follows. Fix the numbers c_i satisfying (1.3) (if they exist) and define $\rho_i = \sum_{i \neq j} |a_{i,j} - c_i|$ for $i \in [n]$. Theorem 1.1 states then, in a form analogous to (1.1), that if λ is an eigenvalue of A with (algebraic) multiplicity at least two, then $\lambda \in \bigcup_{i=1}^n D(a_{i,i} - c_i, \rho_i)$.

2. Earlier results

By definition, the usual or *algebraic* multiplicity of an eigenvalue λ is $k \geq 1$ if $\text{rank}(A - \lambda I) = n - k$. The *geometric* multiplicity of an eigenvalue λ is defined as the dimension of the eigensubspace corresponding to λ . It is clear that the geometric multiplicity is never larger than the algebraic one.

Under some special conditions, the Gershgorin bound has recently been improved for eigenvalues with geometric multiplicity two or more. This is a result Barany and Solymosi [4] followed by Hall and Marsli [10]. Assume A is an $n \times n$ real matrix and let t_i denote the median of the numbers $a_{i,1}, \dots, a_{i,i-1}, 0, a_{i,i+1}, \dots, a_{i,n}$ (here $a_{i,i}$ is replaced by 0) and set $r_i = |t_i| + \sum_{j \neq i} |a_{i,j} - t_i|$. It is easy to see (check Section 3 for an argument) that both t_i and r_i can be determined explicitly. We note that in statistics the quantity $|a_{i,j} - t_i|$ is called the deviation of the sample $a_{i,1}, \dots, a_{i,n}$ at $a_{i,j}$ and so r_i would be the sum of the deviations, cf. [11].

The following result is implicit in [4]; see the inequality in the last lines of page 3 there. A detailed proof appears in Hall and Marsli [10].

THEOREM 2.1. *Assume λ is an eigenvalue of the real matrix A whose geometric multiplicity is at least 2. Then $\lambda \in \bigcup_{i=1}^n D(a_{i,i}, r_i)$.*

A similar result was proved much earlier, in 1954, by Ky Fan and Hoffman [8]. They again require geometric multiplicity, but allow (geometric) multiplicity higher than 2 and the corresponding radius is given in different terms. They say that it was Hadamard who discovered the basic case of Gershgorin's theorem, namely that condition (1.1) implies $\text{rank } A = n$.

We remark here that higher multiplicity might not guarantee a smaller radius in Theorem 2.1. This is shown by J_n , the $n \times n$ all one matrix, because the zero eigenvalue of J_n has (algebraic and geometric) multiplicity $n - 1$ and still, it lies on the boundary of the disk with radius $r_i = 1$ with center $a_{i,i} = 1$.

Our main result, Theorem 1.1, extends Theorem 2.1 to complex matrices and, simultaneously, gets rid of the condition on geometric multiplicity. The proof uses Gershgorin's original theorem and is fairly simple. It also gives a stronger result (Corollary 3.1 below) in the real case. Another target is the extension of Theorem 2.1 in the real case by replacing the smaller Gershgorin disks with slightly larger disks when the underlying matrix is normal, and when it is almost symmetric, see Sections 7 and 8. For instance for normal matrices (the definition is given in Sections 7) we have the following result where r_i is the same as in Theorem 2.1.

THEOREM 2.2. *Assume λ and μ are distinct eigenvalues of a real and normal $n \times n$ matrix A , $n \geq 3$. Then both λ and μ lie in the disk $D(a_{i,i}, \rho_i)$ for some $i \in [n]$ where $\rho_i = r_i + \sqrt{n}|\lambda - \mu|$.*

Checking whether condition (1.3) holds is fairly easy in the real case as we shall see in Section 5. The complex case is more involved and we give some comments on it at the end of that section.

3. The real case

Assume now that A is a real $n \times n$ matrix. For all $i \in [n]$ let t_i^* denote the median of the numbers $a_{i,1}, \dots, a_{i,i-1}, a_{i,i+1}, \dots, a_{i,n}$ and set $r_i^* = \sum_{j \neq i} |a_{i,j} - t_i^*|$. Theorem 1.1 when applied to the real matrix A gives the following result.

COROLLARY 3.1. *Assume A is a real $n \times n$ matrix. If $\lambda \notin \bigcup_{i=1}^n D(a_{i,i}, r_i^*)$, then $\text{rank}(A - \lambda I) \geq n - 1$. In other words, if λ is an eigenvalue of A with algebraic multiplicity two or more, then $\lambda \in \bigcup_{i=1}^n D(a_{i,i}, r_i^*)$.*

This is a stronger version of Theorem 2.1 for two reasons. The first is that λ is a usual (algebraic) eigenvalue of A . The second is that $D(a_{i,i}, r_i^*) \subset D(a_{i,i}, r_i)$ because these two disks have the same centre and

$$(3.1) \quad r_i^* \leq r_i.$$

The simple proof is in Section 5.

In general, bounding the rank is important in linear algebra, such bounds have various applications in different parts of mathematics. There are nice examples of applications to combinatorics in Alon's classical papers [1,2] that provide rank bounds for real matrices where the diagonal elements are larger than other entries in their row, but not large enough for a direct application of Gershgorin's theorem. In Section 6 we give a geometric application of Theorem 1.1 or rather of its special case formulated as Lemma 4.1 in the next section.

4. Proof of Theorem 1.1

We begin with a simple reduction. When $\lambda = 0$, Theorem 1.1 takes the following form.

LEMMA 4.1. *Let A be an $n \times n$ complex matrix. Assume there are numbers $c_i \in \mathbb{C}$ satisfying the inequality*

$$(4.1) \quad |a_{i,i} - c_i| > \sum_{i \neq j} |a_{i,j} - c_i|,$$

for all $i \in [n]$. Then $\text{rank } A \geq n - 1$. \square

The lemma implies the general case by applying it to the matrix $A - \lambda I$. Indeed, the diagonal entries of this matrix are $a_{i,i} - \lambda$ and the off-diagonal ones are simply $a_{i,j}$ so condition (1.3) for A is the same as condition (4.1) for $A - \lambda I$.

The **PROOF** of the lemma is simple and uses diagonally dominant matrices. Let C be the matrix whose every entry in row i is c_i for $i \in [n]$. Then

$\text{rank } C \leq 1$ clearly. Setting $D = A - C$, condition (4.1) says exactly that D is a diagonally dominant matrix. The original Gershgorin theorem shows then that $\text{rank } D = n$. As the rank is subadditive, $\text{rank } D = \text{rank}(A - C) \leq \text{rank } A + \text{rank } C \leq \text{rank } A + 1$ implying that $\text{rank } A \geq \text{rank } D - 1 = n - 1$. \square

We note that when condition (1.3) holds for $c_i = 0$ we get back Gershgorin's circle theorem because then C is the zero matrix that has rank zero.

REMARK. One can go one step further. Define C as above (with parameters c_i) and let E be the $n \times n$ complex matrix with every entry in its j th column equal to e_j . We define $D = A - C - E$, so the entry i, j of D is $a_{i,j} - c_i - e_j$. Again, if the parameters c_i, e_j can be chosen so that D is diagonally dominated, then $\text{rank } A \geq d - 2$.

5. Choosing c_i in the real case

Assume A is a real $n \times n$ matrix and let $b_{i,1} \leq \dots \leq b_{i,n-1}$ be the increasing rearrangement of the $n - 1$ numbers $a_{i,1}, \dots, a_{i,n}$ where $a_{i,i}$ is missing. The function $f(t) = \sum_{i \neq j} |a_{i,j} - t|$ is piecewise linear and convex. It attains its minimum at the median of the numbers $b_{i,j}$. This median is the single point $t_i^* := b_{i,n/2}$ when n is even and can be taken for any point of the middle interval $J := [b_{i,(n-1)/2}, b_{i,(n+1)/2}]$ when n is odd. We remark that in this case, $r_i^* = \sum_{j \neq i} |a_{i,j} - t_i^*|$ is the same no matter what $t \in J$ is chosen for the median. Recall that t_i is the median of the numbers $0, b_{i,1} \leq \dots \leq b_{i,n-1}$ and $r_i = |t_i| + \sum_{j \neq i} |a_{i,j} - t_i|$.

PROOF of the inequality (3.1). Assume first that n is even. Then $t_i^* = b_{i,n/2}$ and t_i , the median of the numbers $0, b_{i,1} \leq \dots \leq b_{i,n-1}$, is an arbitrary element of the middle interval J . It is easy to see that one endpoint of J is equal to t_i^* (no matter where 0 is). So t_i can be taken equal to t_i^* , and then $r_i = |t_i| + \sum_{j \neq i} |a_{i,j} - t_i| = |t_i| + r_i^* \geq r_i^*$, indeed. When n is odd, J is an interval and any point of J can be taken for t_i^* . After adding 0 to the sequence $b_{i,1} \leq \dots \leq b_{i,n-1}$, the median, t_i , is going to be an endpoint of J as one can check directly. So $t_i = t_i^*$ again and $r_i^* \leq r_i$ follows the same way as before. \square

For the application of Theorem 1.1 one has to decide if suitable c_i values exist or not. This is fairly easy for real matrices as explained below.

CLAIM 5.1. *In the real case condition (1.3) is satisfied by some c_i if and only if it is satisfied by $c_i = t_i^*$ when n is even and by one of the endpoints of J when n is odd.*

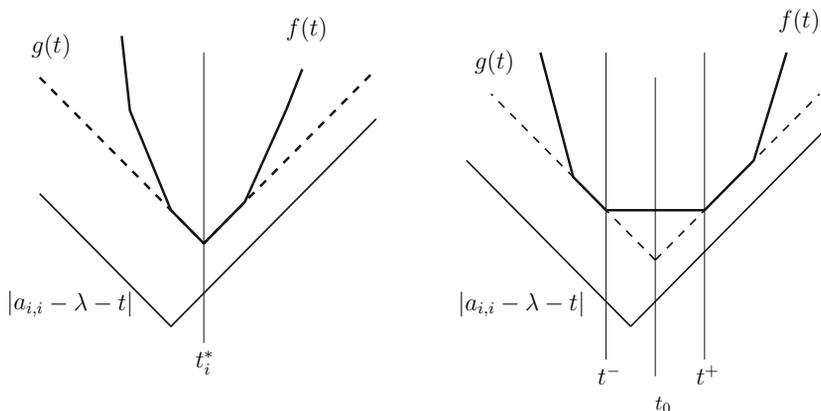


Fig. 1: The functions $f(t)$, $g(t)$ and $|a_{i,i} - \lambda - t|$

PROOF. One direction is easy: there is nothing to check when (1.3) is satisfied at $c_i = t_i^*$ or when it is satisfied by one of the endpoints of J .

For the other direction assume first that n is even and that condition (1.3) is not satisfied at t_i^* , that is $|a_{i,i} - t_i^* - \lambda| \leq f(t_i^*)$. The function $g(t) = |t - t_i^*| + f(t_i^*)$ is a translated copy of the function $t \rightarrow |t|$ and $|a_{i,i} - t - \lambda| \leq g(t)$ for all t follows from $|a_{i,i} - t_i^* - \lambda| \leq f(t_i^*)$. Moreover $g(t)$ has slope $+1$ for $t > t_i^*$ and slope -1 for $t < t_i^*$, while the slope of $f(t)$ is at least 1 for $t > t_i^*$ and at most -1 for $t < t_i^*$; see Figure 1 left. Then $g(t) \leq f(t)$ for all $t \in \mathbb{R}$ because $g(t_i^*) = f(t_i^*)$ and $|a_{i,i} - \lambda - t| \leq f(t)$ for all t follows.

The case of odd n is similar. For simpler notation set $J = [t^-, t^+]$ and $t_0 = \frac{1}{2}(t^- + t^+)$. Suppose that (1.3) is not satisfied at $t = t^-$ and at $t = t^+$. Observe that $f(t^-) = f(t^+)$. Setting $g(t) = |t - t_0| + f(t^+) - \frac{1}{2}(t^+ - t^-)$ we see that $g(t^-) = f(t^-)$ and $g(t^+) = f(t^+)$, see Figure 1 right. Then $|a_{i,i} - \lambda - t| \leq g(t)$ for all t because this inequality holds for $t = t^+$ and $t = t^-$. Next one shows that $g(t) \leq f(t)$ for all $t \in \mathbb{R}$. This holds because for $t > t^+$ the slope of $g(t)$ is 1 and the slope of $f(t)$ is at least one and for $t < t^-$ the slope of $g(t)$ is -1 and that of $f(t)$ is at most -1 . Moreover $f(t)$ is the constant $f(t^+) = f(t^-)$ for $t \in J$, and there $g(t)$ is below this constant. \square

So in the real case in order to check whether condition (1.3) holds and to find the suitable c_i s one has to compare the minimum of $f(t)$ with the value of $|a_{i,i} - \lambda - t|$ at one or two well-defined points.

Finding c_i in the complex case is a different task. We have to decide if the function $t \rightarrow |a_{i,i} - t|$ is below the function $t \rightarrow \sum_{i \neq j} |a_{i,j} - t|$ for all complex numbers t or not. One can check if this holds when $t = a_{i,j}$ for all $j \in [n]$. At every other $t \in \mathbb{C}$ both functions are differentiable so one could,

in principle, decide if the maximum of the function

$$|a_{i,i} - t| - \sum_{i \neq j} |a_{i,j} - t|$$

is positive for any fixed $i \in [n]$ or not. As the real case indicates, a good candidate for c_i is where the function $f(t) = \sum_{j \neq i} |a_{i,j} - t|$ attains its minimum on $t \in \mathbb{C}$. This point can be determined by convex programming. We remark without proof that $\arg \min f(t)$ is a single point unless all the points $a_{i,j}$ ($j \neq i$) are collinear. The latter case is covered by Claim 5.1.

6. A geometric application

We show a geometric application of Lemma 4.1 which is similar to an example of Alon [2] and of Bukh and Cox [5]. The unit ℓ_1 -ball in \mathbb{R}^d is the convex hull of the $2d$ points $\pm e_i$, i.e., the collection of $x \in \mathbb{R}^d$ such that $\|x\|_1 \leq 1$. Here e_i denotes the vector with a 1 in the i th coordinate and 0's elsewhere. The Euclidean distance between e_i and $-e_i$ is 2 (for all i), and the distance between two other vertices is exactly $\sqrt{2}$. What happens if we relax the distance constraints a bit? What is the maximum number of point pairs, (p_i, q_i) in \mathbb{R}^d , to be denoted by k , such that $(p_i - q_i)^2 = 4$ for every $i \in [k]$ and for every pair $i, j \in [k]$, $i \neq j$ we have

$$|(p_i - q_j)^2 - 2| \leq \varepsilon, \quad |(p_i - p_j)^2 - 2| \leq \varepsilon, \quad |(q_i - q_j)^2 - 2| \leq \varepsilon.$$

The answer is that there are at most two extra point pairs if ε is small enough. More precisely

CLAIM 6.1. *If $\varepsilon < 2/(3d + 5)$, then $k \leq d + 2$.*

PROOF. Assume that $k > d + 2$ and consider the first $d + 3$ pairs (p_i, q_i) . The $(d + 2) \times (d + 2)$ matrix M is defined by the dot products $m_{i,j} = (p_i - p_1) \cdot (q_j - p_1)$, for all $2 \leq i, j \leq d + 3$. The rank of M is at most d . The identity

$$(p_i - p_1)^2 + (q_j - p_1)^2 - (p_i - q_j)^2 = 2(p_i - p_1) \cdot (q_j - p_1)$$

(the cosine theorem in trigonometry) and the conditions imply that

$$(2 - \varepsilon) + (2 - \varepsilon) - (2 + \varepsilon) \leq 2(p_i - p_1) \cdot (q_j - p_1) \leq (2 + \varepsilon) + (2 + \varepsilon) - (2 - \varepsilon).$$

Consequently

$$|m_{i,j} - 1| \leq \frac{3}{2}\varepsilon \text{ for } i \neq j \quad \text{and} \quad |m_{i,i} - 1| \geq 1 - \varepsilon \text{ for } 2 \leq i \leq d + 3.$$

We check that Lemma 4.1 applies now with $c_i = 1$ for all $i = 2, 3, \dots, d + 3$. Indeed $|m_{i,i} - c_i| \geq 1 - \varepsilon$ and $\sum_{j \neq i} |m_{i,j} - c_i| \leq (d + 1) \frac{3}{2} \varepsilon$ when $i \neq j$. Thus the inequality $|m_{i,i} - c_i| > \sum_{j \neq i} |m_{i,j} - c_i|$ follows from the condition $\varepsilon < 2/(3d + 5)$ via a simple computation. By Lemma 4.1, $\text{rank } M \geq d + 2 - 1 = d + 1$, contradicting the fact that $\text{rank } M \leq d$. \square

The following construction shows that the result in Claim 6.1 is close to being sharp. Let us choose a d such that there is a Hadamard matrix H_n of order $n = d + 2$. The row vectors of H_n and $-H_n$ form a scaled and rotated ℓ_1 ball in \mathbb{R}^n . The pointset we are going to consider in \mathbb{R}^d consists of the projection of the $2n$ row vectors $\pm \frac{1}{\sqrt{d}} H_n$ by deleting the last two coordinates of every row. The squared distance between non-antipodal points is between $2 - 4/d$ and $2 + 4/d$ while the distance between the $d + 2$ antipodal points is exactly 2.

One can relax the condition $(p_i - q_i)^2 = 4$ to $|(p_i - q_i)^2 - 4| \leq \varepsilon$. In this case, the proof goes along the same steps as above and gives that for $\varepsilon < 2/(3d + 6)$ the maximal number of such p_i, q_i pairs is at most $d + 2$. We omit the details.

7. Proof of Theorem 2.2

Here we show that if two eigenvalues of a real and normal matrix are close, then both of them lie in a smaller Gershgorin disk. A matrix is normal if its eigenvectors belonging to distinct eigenvalues are orthogonal. For instance, a symmetric and real matrix is always normal. The proof method of Theorem 1.1 does not seem to work here and we go back to the original approach (from [4]) that gave Theorem 2.1.

PROOF OF THEOREM 2.2. Recall that here r_i is the same as in Theorem 2.1. Let A be a normal, real, $n \times n$ matrix. Let $v = (v_1, \dots, v_n)$ resp. $w = (w_1, \dots, w_n)$ be the eigenvectors corresponding to λ and μ where $v_i, w_j \in \mathbb{C}$. As $n \geq 3$ there are $\alpha, \beta \in \mathbb{C}$, such that

$$\sum_{i=1}^n \alpha v_i + \beta w_i = 0,$$

and the largest coordinate is one, i.e. $|\alpha v_j + \beta w_j| \leq 1$ for any $j \in [n]$ and $\alpha v_i + \beta w_i = 1$ for some (from now on fixed) $i \in [n]$.

Let $u = \alpha v + \beta w$. Then $Au = \mu \alpha v + \lambda \beta w$ and

$$a_{i,1}u_1 + a_{i,2}u_2 + \dots + a_{i,i} + \dots + a_{i,n}u_n = \mu \alpha v_i + \lambda \beta w_i = \mu + (\lambda - \mu) \beta w_i$$

because $u_i = 1$. Set $S = a_{i,1}u_1 + a_{i,2}u_2 + \dots + a_{i,i} + \dots + a_{i,n}u_n - a_{i,i}$. Then

$$|\mu - a_{i,i}| \leq |S| + |\lambda - \mu| |\beta w_i|.$$

Bounding S comes from Theorem 2 in [4] and, in a slightly more general form, from Lemma 2.4 of [10].

LEMMA 7.1. $|S| \leq r_i$.

The proof is the same as in [4] and we present it at the end of this section.

We bound the error term $|\beta w_i|$ using that A is normal. The two eigenvectors are orthogonal, so we have

$$n \geq |u|^2 = |\alpha v|^2 + |\beta w|^2,$$

which implies that $|\beta w_i| \leq \sqrt{n}$. \square

The PROOF of Lemma 7.1 is simple.

$$\begin{aligned} S &= a_{i,1}u_1 + \dots + a_{i,i-1}u_{i-1} + 0u_i + a_{i,i+1}u_{i+1} + \dots + a_{i,n}u_n \\ &= (a_{i,1} - t)u_1 + \dots + (a_{i,i-1} - t)u_{i-1} \\ &\quad + (0 - t)u_i + (a_{i,i+1} - t)u_{i+1} + \dots + (a_{i,n} - t)u_n \end{aligned}$$

for every $t \in \mathbb{R}$ because $\sum u_i = 0$. As $|u_j| \leq 1$ for all $j \in [n]$ this implies that

$$|S| \leq |a_{i,1} - t| + \dots + |a_{i,i-1} - t| + |0 - t| + |a_{i,i+1} - t| + \dots + |a_{i,n} - t|.$$

As we have seen the minimum of the function $t \rightarrow |a_{i,1} - t| + \dots + |a_{i,n} - t|$ is reached on the median of the numbers $a_{i,1}, \dots, a_{i,i-1}, 0, a_{i,i+1}, \dots, a_{i,n}$. As we have seen, the median is t_i and the minimum is r_i . \square

We only have numerical examples showing that the estimate in Theorem 2.2 is not too weak. For instance, let M be the 7×7 symmetric real matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The eigenvalues of M are $1 \pm \sqrt{15}$, each with multiplicity one, -2 with multiplicity two, and 2 with multiplicity three. (All multiplicities here and in what follows are algebraic.)

- The double eigenvalue, -2 , is an example that Corollary 3.1 is sharp (with $r^* = 0$)

- The two eigenvalues, -2 and $1 - \sqrt{15} \approx -2.873$ are not too far. As Theorem 2.2 states, both of them are in the disk around 1 with a radius

$$3 + \sqrt{7}|-2 - (1 - \sqrt{15})| \approx 5.3.$$

If we change -2 in M to -1.1 , then the two eigenvalues are closer, and the radius of the disk in Theorem 2.2 becomes smaller. In this new matrix, only the first two eigenvalues are different, and they are $(29 \pm \sqrt{5001})/20$. The two eigenvalues $(29 - \sqrt{5001})/20 = -2.086\dots$ and -2 are close to each other. The corresponding radius is $3.22\dots$ and the centre of the disk is at 1 . So the smaller eigenvalue, $-2.086\dots$, is of course inside the disk but quite close to its boundary; their distance is less than 0.14 .

8. Almost symmetric matrices

This section extends the previous results to almost symmetric real matrices.

There are different ways to define almost symmetric matrices. They appear in various contexts in applied linear algebra. For example in [6] and [13] a nearly symmetric matrix is defined as a matrix in which the overwhelming majority of entries are symmetric about its diagonal. In other examples, an almost symmetric matrix M is given as $M = S + E$ where S is a symmetric matrix and E is a small error or “noise”. We can not repeat the arguments on Theorem 2.2 because even small changes can make the eigenvectors far from being orthogonal, like in the example below.

$$M = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 + \varepsilon \end{pmatrix}$$

We will not give a formal definition of what an almost symmetric matrix is. Instead we introduce a parameter, $\Delta(A)$, measuring the symmetry of a matrix A :

$$\Delta(A) = \max_{i \in [n]} \sum_{j=1}^n |a_{i,j} - a_{j,i}|.$$

In this definition the pairs $a_{i,j}$ and $a_{j,i}$ may be far from each other, the parameter $\Delta(A)$ measures how much non-symmetric the corresponding entries in row i and column i are.

With this new matrix parameter, we generalize Theorem 2.2 to arbitrary real matrices. We are going to use the well-known fact that an eigenvector of a real matrix A which belongs to the eigenvalue λ is orthogonal to any eigenvector of A^T which belongs to a different eigenvalue μ .

THEOREM 8.1. *Assume that $A = \{a_{i,j}\}_{i,j=1}^n$ is a real matrix and $n \geq 3$. If λ and μ are two distinct eigenvalues of A , then there is $i \in [n]$ such that both λ and μ lie in the disk $D(a_{i,i}, \rho_i)$ where $\rho_i = r_i + \sqrt{n}(\Delta + |\lambda - \mu|)$ where r_i is the same as before.*

PROOF. Let v be the eigenvector of λ in A and w be the eigenvector of μ in A^T . The two vectors are orthogonal, so they are linearly independent. Then there are $\alpha, \beta \in \mathbb{C}$ such that the coordinates of $\alpha v + \beta w$ add up to zero:

$$\sum_{i=1}^n \alpha v_i + \beta w_i = 0,$$

and the largest norm coordinate (indexed by $i \in [n]$) is one. We assume again that $\alpha v_i + \beta w_i = 1$ for some fixed $i \in [n]$ and $|\alpha v_j + \beta w_j| \leq 1$ for all $j \in [n]$. With these notations we have the system of equations

$$A\alpha v + A^T\beta w = \lambda\alpha v + \mu\beta w = \lambda(\alpha v + \beta w) + (\mu - \lambda)\beta w,$$

Let us consider the i th row

$$\alpha v_1 a_{1,i} + \beta w_1 a_{i,1} + \dots + a_{i,i} + \dots + \alpha v_j a_{j,i} + \beta w_j a_{i,j} + \dots = \lambda + (\mu - \lambda)\beta w_i.$$

From this we have

$$|\lambda - a_{i,i}| \leq \left| \sum_{j \neq i}^n (\alpha v_j + \beta w_j) a_{j,i} - \beta \sum_{j=1}^n w_j (a_{j,i} - a_{i,j}) - (\mu - \lambda)\beta w_i \right|.$$

Using the bound $|\beta w_j| \leq \sqrt{n}$ from the previous proof we have

$$\begin{aligned} |\lambda - a_{i,i}| &\leq \left| \sum_{j \neq i}^n (\alpha v_j + \beta w_j) a_{j,i} - \beta \sum_{j=1}^n w_j (a_{j,i} - a_{i,j}) - (\mu - \lambda)\beta w_i \right| \\ &\leq \left| \sum_{j \neq i}^n (\alpha v_j + \beta w_j) a_{j,i} \right| + \sqrt{n} \sum_{j=1}^n |a_{j,i} - a_{i,j}| + \sqrt{n} |\mu - \lambda| \\ &\leq r_i + \sqrt{n} (\Delta(A) + |\mu - \lambda|). \end{aligned}$$

Here the estimate $\left| \sum_{j \neq i}^n (\alpha v_j + \beta w_j) a_{j,i} \right| \leq r_i$ is the same as in the proof of Theorem 2.2. \square

References

- [1] N. Alon, Problems and results in extremal combinatorics. I, *Discrete Math.*, **273** (2003), 31–53.
- [2] N. Alon, Perturbed identity matrices have high rank: Proof and applications, *Combin. Probab. Comput.*, **18** (2009), 3–15.
- [3] N. Alon and J. Solymosi, Rank of matrices with entries from a multiplicative group, *Int. Math. Res. Not. IMRN* (2022) (to appear).

- [4] I. Bárány and J. Solymosi, Gershgorin disks for multiple eigenvalues of non-negative matrices, in: *A Journey Through Discrete Mathematics*, M. Loeb, J. Nešetřil, R. Thomas, Eds., Springer (Cham, 2017).
- [5] B. Bukh, and C. Cox, Nearly orthogonal vectors and small antipodal spherical codes, *Israel J. Math.*, **238** (2020), 359–388.
- [6] A. H. Berliner and R. A. Brualdi, Nearly symmetric-indecomposable matrices, *Linear Multilinear Algebra*, **63** (2015), 1468–1484.
- [7] Lee DeVille, Optimizing Gershgorin for symmetric matrices, *Linear Algebra Appl.*, **577** (2019), 360–383,
- [8] Ky Fan and A. J. Hoffman, Lower bounds for the rank and location of the eigenvalues of a matrix, in: *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues*, Applied Mathematics Series, no. 39, National Bureau of Standards (1954), pp. 117–130.
- [9] S. Gershgorin, Über die Abgrenzung der Eigenwerte einer Matrix, *Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk*, **6** (1931), 749–754.
- [10] F. J. Hall and R. Marsli, On the location of eigenvalues of real matrices, *Electron. J. Linear Algebra*, **32** (2017), 357–364.
- [11] A. R. Jones, *Probability, Statistics and other Frightening Stuff*, Routledge (London and New York, 2018).
- [12] R. S. Varga, *Gershgorin and his Circles*, Springer Series in Computational Math., vol. 36, Springer (Berlin, 2004).
- [13] Hong Zheng, Sheng Guo Cheng and Quan Sheng Liu, A decomposition procedure for nearly symmetric matrices with applications to some nonlinear problems, *Mech. Res. Commun.*, **37** (2010), 78–84.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.