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A NEW GERŠGORIN-TYPE EIGENVALUE INCLUSION SET*

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Abstract. We give a generalization of a less well-known result of Dashnic and Zusmanovich [2] from 1970, and show how this generalization compares with related results in this area.

Key words. Geršgorin theorem, Brauer Cassini ovals, nonsingularity results.

AMS subject classifications. 15A18, 65F15.

1. Introduction. Our interest here is in nonsingularity results for matrices and their equivalent eigenvalue inclusion sets in the complex plane. As examples of this, we have the famous result of Geršgorin [3]:

THEOREM 1. *any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ and for any eigenvalue λ of A, there is a positive integer k in $N := \{1, 2, \cdots, n\}$ such that*

$$|\lambda - a_{k,k}| \le r_k(A) := \sum_{j\in N\setminus\{k\}} |a_{k,j}|. \tag{1.1}$$

Consequently, if $\sigma(A)$ denotes the collection of all eigenvalues of A, then

$$\sigma(A) \subseteq \Gamma(A) := \bigcup_{i=1}^{n} \Gamma_i(A), \text{ where } \Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \le r_i(A)\}. \tag{1.2}$$

Here, $\Gamma_i(A)$ is the i-th **Geršgorin disk**, and $\Gamma(A)$ is the **Geršgorin set** for the matrix A. The equivalent nonsingularity result for this is

THEOREM 2. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ which is strictly diagonally dominant, i.e.,*

$$|a_{i,i}| > r_i(A) \quad (\text{all } i \in N), \tag{1.3}$$

it follows that A is nonsingular.

Similarly, there is the following nonsingularity result of Ostrowski [5]:

THEOREM 3. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, $n \ge 2$, with*

$$|a_{i,i}| \cdot |a_{j,j}| > r_i(A) \cdot r_j(A) \quad (\text{all } i \ne j \text{ in } N), \tag{1.4}$$

it follows that A is nonsingular.

Its equivalent eigenvalue inclusion set is the following result of Brauer [1]:

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THEOREM 4. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, and for any eigenvalue λ of A, there is a pair of distinct integers i and j in N such that*

$$\lambda \in K_{i,j}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A)\}. \tag{1.5}$$

Consequently,

$$\sigma(A) \subseteq \mathcal{K}(A) := \bigcup_{\substack{i,j \in N \\ i \neq j}} K_{i,j}(A). \tag{1.6}$$

The quantity $K_{i,j}(A)$ of (1.5) is called the (i, j)-th **Brauer Cassini oval**, and $\mathcal{K}(A)$ of (1.6) is called the **Brauer set** for the matrix A. (For further results about these sets, see Varga [6].)

2. New results. To describe our first result here, let S denote a nonempty subset of $N = \{1, 2, \cdots, n\}, n \geq 2$, and let $\overline{S} := N \backslash S$ denote its complement in N. Then, given any matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, split each row sum, $r_i(A)$ from (1.1), into two parts, depending on S and $\overline{S}$, i.e.,

$$\begin{cases} r_i(A) := \displaystyle\sum_{j \in N\backslash\{i\}} |a_{i,j}| = r_i^S(A) + r_i^{\overline{S}}(A), \quad \text{where} \\ r_i^S(A) := \displaystyle\sum_{j \in S\backslash\{i\}} |a_{i,j}|, \text{ and } r_i^{\overline{S}}(A) := \displaystyle\sum_{j \in \overline{S}\backslash\{i\}} |a_{i,j}| \quad (\text{all } i \in N). \end{cases} \tag{2.1}$$

DEFINITION 1. *Given any matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, and given any nonempty subset S of N, then A is an* **S-strictly diagonally dominant matrix** *if*

$$\begin{cases} i) \;\; |a_{i,i}| > r_i^S(A) \;\; (\text{all } i \in S), \quad \text{and} \\ ii) \;\; (|a_{i,i}| - r_i^S(A)) \cdot (|a_{j,j}| - r_j^{\overline{S}}(A)) > r_i^{\overline{S}}(A) \cdot r_j^S(A) \; (\text{all } i \in S, \; \text{all } j \in \overline{S}). \end{cases} \tag{2.2}$$

We note, from (2.2 *i*), that as $|a_{i,i}| - r_i^S(A) > 0$ for all $i \in S$, then on dividing by this term in (2.2 *ii*) gives

$$\left(|a_{j,j}| - r_j^{\overline{S}}(A)\right) > \frac{r_i^{\overline{S}}(A) \cdot r_j^S(A)}{(|a_{i,i}| - r_i^S(A))} \geq 0 \quad (\text{all } j \in \overline{S}),$$

so that we also have

$$|a_{j,j}| - r_j^{\overline{S}}(A) > 0 \quad (\text{all } j \in \overline{S}). \tag{2.3}$$

If $S = N$, so that $\overline{S} = \emptyset$, then the conditions of (2.2 *i*) reduce to $|a_{i,i}| > r_i(A)$ (all $i \in N$), and this is just the familiar statement that A is **strictly diagonally dominant**.

Our first result here is

THEOREM 5. *Let S be a nonempty subset of N, and let $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, be S-strictly diagonally dominant. Then, A is nonsingular.*

Proof. If $S = N$, then, as we have seen, A is strictly diagonally dominant, and thus nonsingular from Theorem 2. Next, we assume that S is a nonempty subset of N with $\overline{S} \neq \emptyset$.

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The idea of the proof is to construct a positive diagonal matrix W such that AW is strictly diagonally dominant. Now, define W as $W = \mathrm{diag}[w_1, w_2, \cdots, w_n]$, where

$$w_k := \begin{cases} \gamma, & \text{for all } k \in S, \text{ where } \gamma > 0, \text{ and} \\ 1, & \text{for all } k \in \overline{S}. \end{cases}$$

It then follows that $AW := [\alpha_{i,j}] \in \mathbb{C}^{n \times n}$ has its entries given by

$$\alpha_{i,j} := \begin{cases} \gamma a_{i,j}, & \text{if } j \in S, \text{ all } i \in N, \text{ and} \\ a_{i,j}, & \text{if } j \in \overline{S}, \text{ all } i \in N. \end{cases}$$

Then, the row sums of AW are, from (2.1), just

$$r_\ell(AW) = r_\ell^S(AW) + r_\ell^{\overline{S}}(AW) = \gamma r_\ell^S(A) + r_\ell^{\overline{S}}(A) \ \ (\text{all } \ell \in N),$$

and AW is then strictly diagonally dominant if

$$\begin{cases} \gamma |a_{i,i}| > \gamma r_i^S(A) + r_i^{\overline{S}}(A) \ \ (\text{all } i \in S), \text{ and} \\ |a_{j,j}| > \gamma r_j^S(A) + r_j^{\overline{S}}(A) \ \ (\text{all } j \in \overline{S}). \end{cases}$$

The above inequalities can be also expressed as

$$\begin{cases} i)\ \gamma(|a_{i,i}| - r_i^S(A)) > r_i^{\overline{S}}(A) \ \ (\text{all } \ i \in S), \text{ and} \\ ii)\ \ |a_{j,j}| - r_j^{\overline{S}}(A) > \gamma r_j^S(A) \ \ (\text{all } j \in \overline{S}), \end{cases} \tag{2.4}$$

which, upon division, can be further reduced to

$$\frac{r_i^{\overline{S}}(A)}{|a_{i,i}| - r_i^S(A)} < \gamma \ (\text{all } i \in S), \text{ and } \gamma < \frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} \ (\text{all } j \in \overline{S}), \tag{2.5}$$

where the final fraction in (2.5) is defined to be $+\infty$ if $r_j^S(A) = 0$ for some $j \in \overline{S}$. The inequalities of (2.4) will all be satisfied if there is a $\gamma > 0$ for which

$$0 \le B_1 := \max_{i \in S} \frac{r_i^{\overline{S}}(A)}{|a_{i,i}| - r_i^S(A)} < \gamma < \min_{j \in \overline{S}} \frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} =: B_2. \tag{2.6}$$

But since (2.2 *ii*) exactly gives that $B_2 > B_1$, then, for any $\gamma > 0$ with $B_1 < \gamma < B_2$, AW is strictly diagonally dominant and hence nonsingular. Then, as W is nonsingular, so is A. □

As is now familiar, the nonsingularity in Theorem 2 then gives, by negation, the following equivalent eigenvalue inclusion set in the complex plane.

THEOREM 6. *Let S be any nonempty subset of $N := \{1, 2, \cdots, n\}$, $n \ge 2$, with $\overline{S} := N \backslash S$. Then, for any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, define the Geršgorin-type disks*

$$\Gamma_i^S(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \le r_i^S(A)\} \ (\text{any } i \in S), \tag{2.7}$$

and the sets

$$V_{i,j}^S(A) := \{z \in \mathbb{C} : (|z - a_{i,i}| - r_i^S(A)) \cdot (|z - a_{j,j}| - r_j^{\overline{S}}(A)) \le r_i^{\overline{S}}(A) \cdot r_j^S(A)\}, \tag{2.8}$$

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(*any* $i \in S$, *any* $j \in \overline{S}$). *Then,*

$$\sigma(A) \subseteq C^S(A) := \left(\bigcup_{i \in S} \Gamma_i^S(A) \right) \cup \left(\bigcup_{i \in S, j \in \overline{S}} V_{i,j}^S(A) \right). \tag{2.9}$$

We remark that Dashnic and Zusmanovich [2] obtained the result of Theorem 5 in the special case that the set S is a singleton, i.e., $S_i := \{i\}$ for some $i \in N$. In this case, we define the associated set, from Theorem 6, as the set $\mathcal{D}_i(A)$, so that, from (2.7) and (2.8),

$$\mathcal{D}_i(A) = \Gamma_i^{S_i}(A) \cup \left(\bigcup_{j \in N \setminus \{i\}} V_{i,j}^{S_i}(A) \right). \tag{2.10}$$

Now, $r_i^{S_i}(A) = 0$ from (2.1) so that $\Gamma_i^{S_i}(A) = \{a_{i,i}\}$ from (2.7). Moreover, we also have, from (2.8) in this case that, for all $j \neq i$ in N,

$$V_{i,j}^{S_i}(A) = \{z \in \mathbb{C} : |z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|\ ) \leq r_i(A) \cdot |a_{j,i}|\}. \tag{2.11}$$

But as $z = a_{i,i}$ is necessarily contained in $V_{i,j}^{S_i}(A)$ for all $j \neq i$, we can simply write from (2.11) that

$$\mathcal{D}_i(A) = \bigcup_{j \in N \setminus \{i\}} V_{i,j}^{S_i}(A) \quad (\text{any } i \in N). \tag{2.12}$$

This shows that $D_i(A)$ is determined from $(n-1)$ sets $V_{i,j}^{S_i}(A)$, plus the added information from (2.1) on the partial row sums of A. The associated Geršgorin set $\Gamma(A)$, from (1.2), is determined from n disks and the associated Brauer set $\mathcal{K}(A)$, from (1.6) is determined from $\binom{n}{2}$ Cassini ovals. These sets are compared in the next section.

3. Comparisons with other eigenvalue inclusion sets.

We first establish the new result of

THEOREM 7. *For any* $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, *and for any* $i \in N$, *consider* $\mathcal{D}_i(A)$ *of* (2.12). *Then* (cf. (1.2)),

$$\mathcal{D}_i(A) \subseteq \Gamma(A), \tag{3.1}$$

and for $n = 2$, *and for all* $A = [a_{i,j}] \in \mathbb{C}^{2 \times 2}$, *we have (cf. (1.5) and (1.6))*

$$\mathcal{D}_1(A) = \mathcal{D}_2(A) = \mathcal{K}(A) = K_{1,2}(A). \tag{3.2}$$

But, for any $n \geq 3$ *and for any* $i \in N$, *there is a matrix* $\tilde{F}$ *in* $\mathbb{C}^{n \times n}$ *for which*

$$\mathcal{D}_i(\tilde{F}) \nsubseteq \mathcal{K}(\tilde{F}) \text{ and } \mathcal{K}(\tilde{F}) \nsubseteq \mathcal{D}_i(\tilde{F}). \tag{3.3}$$

Proof. To establish (3.1), fix some $i \in N$ and consider any $z \in \mathcal{D}_i(A)$. Then from (2.12), there is a $j \neq i$ such that $z \in V_{i,j}^{S_i}(A)$, i.e., from (2.11),

$$|z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|) \leq r_i(A) \cdot |a_{j,i}|. \tag{3.4}$$

If $z \notin \Gamma(A)$, then $|z - a_{k,k}| > r_k(A)$ for *all* $k \in N$, so that $|z - a_{i,i}| > r_i(A) \geq 0$, and $|z - a_{j,j}| > r_j(A) \geq 0$. Thus, the left part of (3.4) satisfies

$$|z - a_{i,i}| \cdot (|z - a_{j,j}| - r_j(A) + |a_{j,i}|) > r_i(A) \cdot |a_{j,i}|,$$

which contradicts the inequality in (3.4). Thus, $z \in \Gamma(A)$ for each $z \in \mathcal{D}_i(A)$, which establishes (3.1).

Next, to establish (3.2), it can be easily seen from (1.5)-(1.6) and (2.11)-(2.12) that (3.2) is valid for any $A = [a_{i,j}] \in \mathbb{C}^{2\times 2}$.

Finally, to establish (3.3), consider first the specific 3×3 matrix E of

$$E = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & i & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{3.5}$$

Then, it can be verified that

$$\begin{aligned} \Gamma(E) = \; & \{z \in \mathbb{C} : |z-1| \leq 1\} \cup \{z \in \mathbb{C} : |z-i| \leq 1\} \cup \{z \in \mathbb{C} : |z+1| \leq 1\}, \\ \mathcal{K}(E) = \; & \{z \in \mathbb{C} : |z-1| \cdot |z-i| \leq 1\} \cup \{z \in \mathbb{C} : |z-i| \cdot |z+1| \leq 1\} \\ & \cup \{z \in \mathbb{C} : |z-1| \cdot |z+1| \leq 1\}, \\ \mathcal{D}_1(E) = \; & \{z \in \mathbb{C} : |z-1| \cdot (|z-i| - 1) \leq 0\} \cup \{z \in \mathbb{C} : |z-1| \cdot (|z+1| - 1) \leq 0\}. \end{aligned}$$

It is interesting to note that $\mathcal{D}_1(E)$ reduces to the union of the two disks $\{z \in \mathbb{C} : |z - i| \leq 1\}$ and $\{z \in \mathbb{C} : |z + 1| \leq 1\}$, *and* the single point $z = 1$. These above three sets are shown in Fig. 3.1, where we see that the special case $i = 1$ and $n = 3$ of (3.3) is valid.

To establish (3.3) in general, let $n > 3$, and consider the matrix F in $\mathbb{C}^{n\times n}$ which is obtained by adding $n - 3$ rows of zeros beneath the matrix E of (3.5) and $n - 3$ columns of zeros to the right of E, so that E becomes the upper 3×3 principal submatrix of F. From the structure of F, it is not difficult to show that (3.3) holds for F in the case $i = 1$, i.e.,

$$\mathcal{D}_1(F) \nsubseteq \mathcal{K}(F) \text{ and } \mathcal{K}(F) \nsubseteq \mathcal{D}_1(F).$$

But, given any $i \in N$, there is a suitable $n \times n$ permutation matrix P such that if $\tilde{F} := P^T F P$, then

$$\mathcal{D}_i(\tilde{F}) \nsubseteq \mathcal{K}(\tilde{F}) \text{ and } \mathcal{K}(\tilde{F}) \nsubseteq \mathcal{D}_i(\tilde{F}),$$

completing the proof of Theorem 7. ☐

Next, it is evident from (2.9) of Theorem 6 that, for any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$,

$$\sigma(A) \subseteq \mathcal{D}_i(A) \quad (\text{all } i \in N),$$

so that

$$\sigma(A) \subseteq \mathcal{D}(A) := \bigcap_{i \in N} \mathcal{D}_i(A). \tag{3.6}$$

Now, as each $\mathcal{D}_i(A)$, from (2.12), depends on $(n - 1)$ oval-like sets $V_{i,j}^{S_i}(A)$, it follows that $\mathcal{D}(A)$ of (3.6) is determined from $n(n - 1)$ oval-like sets $V_{i,j}^{S_i}(A)$, which is *twice* the number of Cassini ovals, namely $\binom{n}{2}$, which determine the Brauer set $\mathcal{K}(A)$. This suggests, perhaps, that $\mathcal{D}(A) \subseteq \mathcal{K}(A)$. This inclusion is true, and this new result is established in

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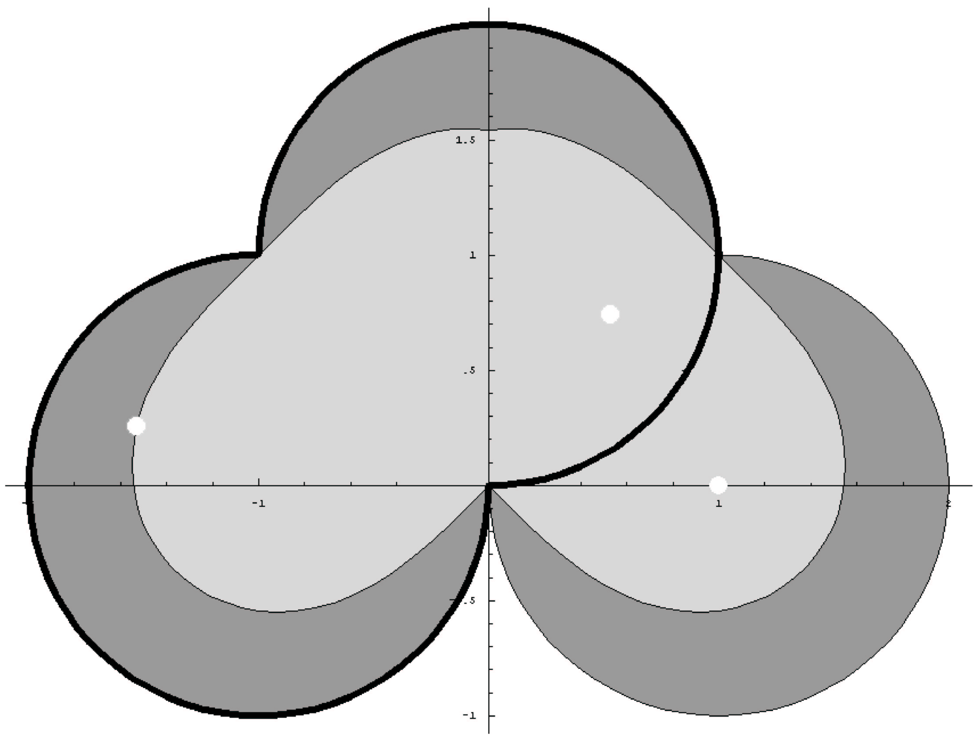

FIG. 3.1. *The sets* $\Gamma(E)$ *(shaded dark gray),* $\mathcal{K}(E)$ *(shaded light gray),* $\mathcal{D}_1(E)$ *(two disks with the bold boundary and the point* $z = 1$*) for the matrix* E *of (3.5). The white dots are the eigenvalues of* E.

THEOREM 8. *For any* $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$*, the associated sets* $\mathcal{D}(A)$*, of (3.6), and* $\mathcal{K}(A)$*, of (1.6), satisfy*

$$\mathcal{D}(A) \subseteq \mathcal{K}(A). \tag{3.7}$$

Proof. First, we observe, from (3.1), that as $\mathcal{D}_i(A) \subseteq \Gamma(A)$ for each $i \in N$, then $\mathcal{D}(A)$, as defined in (3.6), evidently satisfies

$$\mathcal{D}(A) \subseteq \Gamma(A). \tag{3.8}$$

To establish (3.7), consider any $z \in \mathcal{D}(A)$ so that, for each $i \in N$, $z \in \mathcal{D}_i(A)$. Hence, from (2.12), for each $i \in N$, there is $j \in N \backslash \{i\}$ so that $z \in V_{i,j}^{S_i}(A)$, i.e. the inequality of (3.4) is valid. But from (3.8), $\mathcal{D}(A) \subseteq \Gamma(A)$ implies that there is a $k \in N$ with $|z - a_{k,k}| \leq r_k(A)$. For this index k, there is a $t \in N\backslash\{k\}$ such that $z \in V_{k,t}^{S_i}(A)$, i.e.,

$$|z - a_{k,k}| \left(|z - a_{t,t}| - r_t(A) + |a_{t,k}|\right) \leq r_k(A) \cdot |a_{t,k}|.$$

This can be rewritten as

$$\begin{aligned} |z - a_{k,k}| \cdot |z - a_{t,t}| &\leq |z - a_{k,k}| \cdot (r_t(A) - |a_{t,k}|) + r_k(A) \cdot |a_{t,k}| \\ &\leq r_k(A)(r_t(A) - |a_{t,k}|) + r_k(A) \cdot |a_{t,k}| = r_k(A) \cdot r_t(A), \end{aligned}$$

that is,

$$|z - a_{k,k}| \cdot |z - a_{t,t}| \leq r_k(A) \cdot r_t(A).$$

Hence, from (1.5) and (1.6), $z \in K_{k,t}(A) \subseteq \mathcal{K}(A)$. As this is true for each $z \in \mathcal{D}(A)$, then $\mathcal{D}(A) \subseteq \mathcal{K}(A)$. $\square$

We remark that the set $\mathcal{D}(A)$ of (3.5) was also considered in Dashnic and Zusmanovich [2], but with no comparisons with $\Gamma(A)$ or $\mathcal{K}(A)$.

It is interesting also to mention that Huang [4] similarly breaks $N = \{1, 2, \cdots, n\}$ into disjoint subsets S and $\overline{S}$, but assumes a variant of the inequalities of (2.2). Now, if $S = \{i_1, i_2, \cdots, i_k\}$, then $A_{S,S} := [a_{i_j,i_\ell}]$ (all i_j, i_ℓ in S) is its associated $k \times k$ principal submatrix of A, whose associated **comparison matrix** is given by

$$\mathcal{M}(A_{S,S}) := \begin{bmatrix} +|a_{i_1,i_1}| & -|a_{i_1,i_2}| & \cdots & -|a_{i_1,i_k}| \\ -|a_{i_2,i_1}| & +|a_{i_2,i_2}| & \cdots & -|a_{i_2,i_k}| \\ \vdots & & & \vdots \\ -|a_{i_k,i_1}| & -|a_{i_k,i_2}| & \cdots & +|a_{i_k,i_k}| \end{bmatrix}, \tag{3.9}$$

and it is assumed by Huang that $\mathcal{M}(A_{S,S})$ is a **nonsingular M-Matrix** (or equivalently, that $A_{S,S}$ is a nonsingular H-matrix), with the additional assumption (in analogy with (2.6)) that if $\mathbf{r}^{\overline{S}}(A) := [r_{i_1}^{\overline{S}}(A), r_{i_2}^{\overline{S}}(A), \cdots, r_{i_k}^{\overline{S}}(A)]^T$, then

$$||\mathcal{M}^{-1}(A_{S,S}) \cdot \mathbf{r}^{\overline{S}}(A)||_\infty < B_2 := \min_{j \in \overline{S}} \left(\frac{|a_{j,j}| - r_j^{\overline{S}}(A)}{r_j^S(A)} \right), \tag{3.10}$$

where B_2 is defined in (2.6). We note that our earlier assumption in (2.2 *i*) makes the associated principal submatrix $A_{S,S}$ a strictly diagonally dominant matrix, so that $\mathcal{M}(A_{S,S})$ in our case is necessarily a nonsingular M-matrix.

The result of Huang [4] is more general than the result of our Theorem 5, but it comes with the added expense of having to explicitly determine $\mathcal{M}^{-1}(A_{S,S})$ for use in (3.10).

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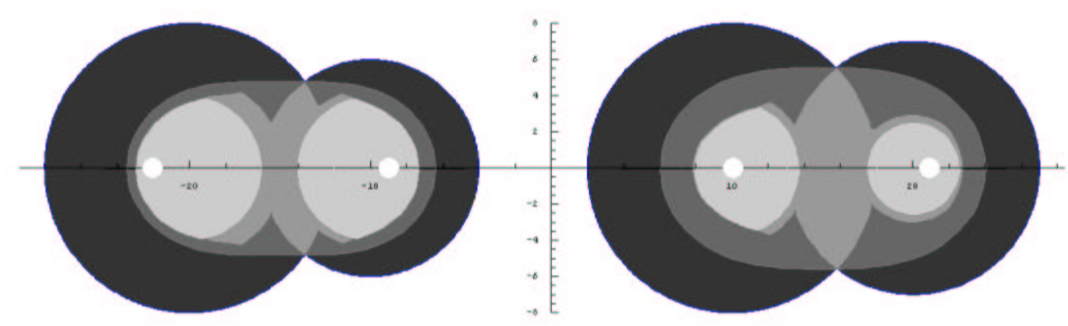

FIG. 4.1. *Considered localization sets referring to the matrix G.*

4. Numerical example. Finally, we give an example of possible improvement in the eigenvalue localization for a given matrix. For the matrix

$$G = \begin{bmatrix} 10 & 0 & 3 & 5 \\ 0 & -10 & 2 & 4 \\ 2 & 5 & 20 & 0 \\ 4 & 4 & 0 & -20 \end{bmatrix},$$

Fig. 4.1 shows the sets $\Gamma(G)$, $\mathcal{K}(G)$, $\mathcal{D}(G)$and $\mathcal{C}(G) := \bigcap_{S \subset N} C^S(G)$ of (1.2), (1.6), (3.6) and (2.9) respectively, shaded decreasingly. Exact eigenvalues are marked by white dots.

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