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Between Geršgorin and minimal Geršgorin sets☆

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Abstract

The eigenvalues of a given matrix A can be localized by the well-known Geršgorin theorem: they belong to the Geršgorin set, which is the union of the Geršgorin disks (each of them is a simple function of the matrix entries). By applying the same theorem to a similar matrix $X^{-1}AX$, a new inclusion set can be obtained. Taking the intersection over X, being a (positive) diagonal matrix, will lead us to the minimal Geršgorin set, defined by Varga [R.S. Varga, Geršgorin and His Circles, Springer Series in Computational Mathematics, vol. 36, 2004], but this set is not easy to calculate. In this paper we will take the intersection over some special structured matrices X and show that this intersection can be expressed by the same formula as the eigenvalue inclusion set $C^S(A)$ in [L.J. Cvetković, V. Kostić, R. Varga, A new Geršgorin-type eigenvalue inclusion set, ETNA 18 (2004) 73–80].

MSC: 65F15

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1. Introduction

The well-known Geršgorin set $\Gamma(A)$ containing all eigenvalues of the matrix $A=[a_{ij}]\in \mathbf{C}^{n,n}$, is given by

$$\Gamma(A) := \bigcup_{i\in N} \Gamma_i(A),$$

$$\Gamma_i(A) := \{z\in \mathbf{C} : |z-a_{ii}|\leqslant r_i(A)\},\quad i\in N:=\{1,2,\ldots,n\},$$

where

$$r_i(A) := \sum_{j\neq i} |a_{ij}|,\quad i\in N.$$

Using a nonsingular matrix X one can obtain another set $\Gamma^X(A) := \Gamma(X^{-1}AX)$ which also contains all the eigenvalues of the matrix A. Furthermore, denoting by $\mathscr{B}$ an arbitrary set of nonsingular matrices, we get the new eigenvalue

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inclusion area for the matrix A:

$$\Gamma^{\mathscr{B}}(A)=\bigcap_{X\in\mathscr{B}}\Gamma^{X}(A).$$

We will call this set the minimal Geršgorin set with respect to $\mathscr{B}$.

The case of $\mathscr{B}=\mathscr{R}$, where $\mathscr{R}$ is the set of all diagonal nonsingular matrices was studied in detail by Varga, [3], where a lot of theoretical properties of this set were proved. Generally, how to find this set is a very hard problem, since there is no explicit "calculable" formula of it. This is the reason for considering the minimal Geršgorin sets with respect to some special subsets of $\mathscr{R}$, for which a formula depending on the matrix entries only could be found. The first step in this direction was done by Dashnic and Zusmanovich [2], where $\mathscr{B}=\mathscr{D}$ and $\mathscr{D}$ is the union of all diagonal matrices, whose diagonal entries are equal to 1, except one, which is an arbitrary positive number. The corresponding minimal Geršgorin set (with respect to $\mathscr{D}$) is described by

$$\Gamma^{\mathscr{D}}(A):=\bigcap_{i\in N}\bigcup_{j\in N,j\neq i}D_{ij}(A),$$

$$D_{ij}(A)=\{z\in\mathbf{C}:|z-a_{ii}|(|z-a_{jj}|-r_j(A)+|a_{ji}|)\leqslant r_i(A)|a_{ji}|\},\quad i,j\in N,\quad j\neq i.$$

In this paper we will go one step more... . In our case $\mathscr{B}=\mathscr{W}$ will be the set of all diagonal matrices whose diagonal entries are either 1 or x, where x is an arbitrary positive number. Namely,

$$\mathscr{W}=\bigcup_{S\subset N}\mathscr{W}^{S},$$

$$\mathscr{W}^{S}=\{X=diag(x_1,x_2,\ldots,x_n):x_i=1\text{ for }i\in S\text{ and }x_i=x>0\text{ otherwise}\}.$$

2. Preliminaries

Lemma 1. *Let $S\subset N$ be a nonempty proper subset of indices, $0<\alpha<\beta<\infty$, and let G_i, $i\in N$, be the functions satisfying*:

- *for any $x\in[\alpha,\beta]$, $G_i(x)$ is a compact set in $\mathbf{C}$ for each $i\in N$;*
- *for every $i\in S$ and for every x and x' in $[\alpha,\beta]$ with $x\leqslant x'$, then $G_i(x)\subset G_i(x')$;*
- *for every $j\in\overline{S}:=N\backslash S$ and for every x and x' in $[\alpha,\beta]$ with $x\leqslant x'$, then $G_j(x')\subset G_j(x)$.*

Then

$$\bigcap_{\alpha\leqslant x\leqslant\beta}\bigcup_{i\in N}G_i(x)=\bigcup_{i\in S,j\in\overline{S}}\bigcap_{\alpha\leqslant x\leqslant\beta}(G_i(x)\cup G_j(x)).$$

Proof. It is obvious that

$$\bigcup_{i\in S,j\in\overline{S}}\bigcap_{\alpha\leqslant x\leqslant\beta}(G_i(x)\cup G_j(x))\subset\bigcap_{\alpha\leqslant x\leqslant\beta}\bigcup_{i\in N}G_i(x),$$

so it remains to prove the opposite inclusion. Let

$$z\in\bigcap_{\alpha\leqslant x\leqslant\beta}\bigcup_{i\in N}G_i(x).$$

This means that for each $x\in[\alpha,\beta]$ there exists $k\in N$ such that $z\in G_k(x)$.

Suppose that

$$z \notin \bigcup_{i\in S,\, j\in \overline{S}} \bigcap_{\alpha \leqslant x \leqslant \beta} (G_i(x) \cup G_j(x)),$$

i.e., for all $i \in S$, $j \in \overline{S}$ there exists $x_{ij} \in [\alpha, \beta]$ such that $z \notin G_i(x_{ij})$ and $z \notin G_j(x_{ij})$.

For all $i \in S$ we define

$$\gamma_i := \max_{j\in \overline{S}} x_{ij} \quad \text{and} \quad \gamma := \min_{i\in S} \gamma_i.$$

Since for every $i \in S$ we have $\gamma_i \geqslant x_{ij}$ (for all $j \in \overline{S}$), using the fact that G_j (for all $j \in \overline{S}$) are nonincreasing functions, we conclude that $z \notin G_j(\gamma_i)$, and consequently $z \notin G_j(\gamma)$ (for all $j \in \overline{S}$).

Hence, there exists $k \in S$ such that $z \in G_k(\gamma)$. The function G_k is a nondecreasing function, so because of $\gamma_k \geqslant \gamma$, we get $z \in G_k(\gamma_k)$. But, $\gamma_k = x_{kj}$ for some $j \in \overline{S}$, which contradicts our assumption and completes the proof. □

Lemma 2. *Let* $0 < \alpha < \beta < \infty$, $a_1, a_2 \in \mathbf{C}$, $r_1, r_2, b_1, b_2 \geqslant 0$. *If for* $x \in [\alpha, \beta]$

$$K_1(x) = \{z \in \mathbf{C} : |z - a_1| \leqslant r_1 + b_1 x\} \quad \textit{and} \quad K_2(x) = \left\{z \in \mathbf{C} : |z - a_2| \leqslant r_2 + \frac{b_2}{x}\right\}$$

are families of circles satisfying $K_1(\alpha) \subset K_2(\alpha)$ *and* $K_2(\beta) \subset K_1(\beta)$, *then*

$$\bigcap_{\alpha \leqslant x \leqslant \beta} (K_1(x) \cup K_2(x)) = \{z \in \mathbf{C} : (|z - a_1| - r_1)(|z - a_2| - r_2) \leqslant b_1 b_2\}.$$

Proof. It is sufficient to prove that

$$\bigcap_{\alpha \leqslant x \leqslant \beta} (K_1(x) \cup K_2(x)) \subset \{z \in \mathbf{C} : (|z - a_1| - r_1)(|z - a_2| - r_2) \leqslant b_1 b_2\},$$

since the opposite inclusion holds trivially.

Suppose that

$$z \in \bigcap_{\alpha \leqslant x \leqslant \beta} (K_1(x) \cup K_2(x)).$$

Then $z \in (K_1(\alpha) \cup K_2(\alpha)) = K_2(\alpha)$ and $z \in (K_1(\beta) \cup K_2(\beta)) = K_1(\beta)$.

If $z \in K_1(\alpha)$, from

$$|z - a_1| - r_1 \leqslant b_1 \alpha \quad \text{and} \quad |z - a_2| - r_2 \leqslant \frac{b_2}{\alpha}$$

it follows that

$$(|z - a_1| - r_1)(|z - a_2| - r_2) \leqslant b_1 b_2$$

and the proof is completed.

On the contrary, if $z \notin K_1(\alpha)$, we have

$$|z - a_1| - r_1 \leqslant b_1 \beta \quad \text{and} \quad |z - a_1| - r_1 > b_1 \alpha,$$

so there exists $\gamma \in (\alpha, \beta]$, such that $|z - a_1| - r_1 = b_1 \gamma$, meaning $z \in K_1(\gamma)$. However, $z \in K_2(\gamma)$, too, since the opposite statement

$$|z - a_2| - r_2 > \frac{b_2}{\gamma}$$

leads to the conclusion that there exists an $\varepsilon > 0$ such that

$$|z - a_2| - r_2 > \frac{b_2}{\gamma - \varepsilon},$$

i.e., $z \notin K_2(\gamma - \varepsilon)$, which contradicts the fact that $z \notin K_1(\gamma - \varepsilon)$!

So, there always exists at least one $x \in [\alpha, \beta]$ such that $z \in K_1(x)$ and $z \in K_2(x)$ which ends the proof. □

3. New result

For a given $S \subset N$ and each $i \in N$ we will "split" the deleted row sum $r_i(A)$ into two components:

$$r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A) \quad \text{where } r_i^S(A) := \sum_{j \in S, j \neq i} |a_{ij}|.$$

Then we form the following sets, depending on S:

$$\mathscr{V}_{ij}^S(A) := \{z \in \mathbf{C} : (|z - a_{ii}| - r_i^S(A))(|z - a_{jj}| - r_j^{\bar{S}}(A)) \leqslant r_i^{\bar{S}}(A) r_j^S(A)\}, \quad i \in S, \quad j \in \bar{S},$$

$$\Gamma_i^S(A) := \{z \in \mathbf{C} : |z - a_{ii}| \leqslant r_i^S(A)\}, \quad i \in S,$$

$$\Gamma_j^{\overline{S}}(A) := \{z \in \mathbf{C} : |z - a_{jj}| \leqslant r_j^{\overline{S}}(A)\}, \quad j \in \overline{S}.$$

Finally, we denote

$$\mathscr{C}(A) := \bigcap_{S \subset N} \left[\left(\bigcup_{i \in S, j \in \bar{S}} \mathscr{V}_{ij}^S(A) \right) \cup \left(\bigcup_{i \in S} \Gamma_i^S(A) \right) \cup \left(\bigcup_{j \in \overline{S}} \Gamma_j^{\overline{S}}(A) \right) \right].$$

One can easily show that the set $\mathscr{C}(A)$ is exactly the same as it appears in [1]. Here, we will show that this set is a sort of minimal Geršgorin set, too. Namely, we prove the following:

Theorem 1.

$$\Gamma^{\mathscr{W}}(A) = \mathscr{C}(A).$$

Proof. For a given subset S let us denote

$$\mathscr{C}^S(A) := \left[\left(\bigcup_{i \in S, j \in \bar{S}} \mathscr{V}_{ij}^S(A) \right) \cup \left(\bigcup_{i \in S} \Gamma_i^S(A) \right) \cup \left(\bigcup_{j \in \overline{S}} \Gamma_j^{\overline{S}}(A) \right) \right].$$

Then, obviously,

$$\mathscr{C}(A) := \bigcap_{S \subset N} \mathscr{C}^S(A)$$

and it is sufficient to prove that

$$\Gamma^{\mathscr{W}^S}(A) = \mathscr{C}^S(A)$$

for all subsets S. So, from now on, we will fix the subset S of indices, and without loss of generality, we will suppose that S is a nonempty proper subset of N.

First of all, let us find the expression for the set $\Gamma^{\mathscr{W}^S_{\alpha,\beta}}(A)$, where

$$\mathscr{W}^S_{\alpha,\beta} := \{X \in \mathscr{W}^S : \alpha \leqslant x \leqslant \beta\}.$$

By its definition,

$$\Gamma^{\mathscr{W}^S_{\alpha,\beta}}(A) = \bigcap_{\alpha \leqslant x \leqslant \beta} \left[\left(\bigcup_{i \in S} V_i^x \right) \cup \left(\bigcup_{j \in \overline{S}} W_j^x \right) \right],$$

where

$$V_i^x := \{z \in \mathbf{C} : (|z - a_{ii}| - r_i^S(A)) \leqslant x r_i^{\overline{S}}(A)\},$$

$$W_j^x := \left\{ z \in \mathbf{C} : (|z - a_{jj}| - r_j^{\overline{S}}(A)) \leqslant \frac{1}{x} r_j^S(A) \right\}.$$

From Lemma 1 it follows that

$$\Gamma^{\mathscr{W}^S_{\alpha,\beta}}(A) = \bigcup_{i \in S, j \in \overline{S}} \bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x]. \tag{1}$$

Let us show that for sufficiently small α and sufficiently large β,

$$\bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x] = \mathscr{V}_{ij}^S(A) \cup \Gamma_i^S(A) \cup \Gamma_j^{\overline{S}}(A) \cup V_i^\alpha \cup W_j^\beta. \tag{2}$$

By the careful inspection, one can see that it is sufficient to prove the inclusion

$$\bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x] \subset \mathscr{V}_{ij}^S(A) \cup \Gamma_i^S(A) \cup \Gamma_j^{\overline{S}}(A) \cup V_i^\alpha \cup W_j^\beta.$$

- If $r_i^{\overline{S}}(A) \neq 0$ and $r_j^S(A) \neq 0$, then for sufficiently small α and sufficiently large β it holds that $W_j^\beta \subset V_i^\beta$ and $V_i^\alpha \subset W_j^\alpha$, so by the Lemma 2 we have

  $$\bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x] = \mathscr{V}_{ij}^S(A).$$

- If $r_i^{\overline{S}}(A) = 0$ and $r_j^S(A) = 0$, then for all $0 < \alpha < \beta < \infty$ and for every $x \in [\alpha, \beta]$ $V_i^x = \Gamma_i^S(A)$ and $W_j^x = \Gamma_j^{\overline{S}}(A)$.
- If $r_i^{\overline{S}}(A) = 0$ and $r_j^S(A) \neq 0$, then for all $0 < \alpha < \beta < \infty$ and for every $x \in [\alpha, \beta]$ $V_i^x = \Gamma_i^S(A)$.
  Now, $\bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x] \subset [V_i^\beta \cup W_j^\beta] = \Gamma_i^S(A) \cup W_j^\beta$.
- Finally, if $r_i^{\overline{S}}(A) \neq 0$ and $r_j^S(A) = 0$, we obtain $\bigcap_{\alpha \leqslant x \leqslant \beta} [V_i^x \cup W_j^x] \subset \Gamma_j^{\overline{S}} \cup V_i^\alpha$.

Hence, we have proved that for sufficiently small α and sufficiently large β, the equality (2) holds. Now, from (1) we get

$$\Gamma^{\mathscr{W}^S_{\alpha,\beta}}(A) = \bigcup_{i \in S, j \in \overline{S}} \mathscr{V}_{ij}^S(A) \cup \Gamma_i^S(A) \cup \Gamma_j^{\overline{S}}(A) \cup V_i^\alpha \cup W_j^\beta.$$

To complete the proof it remains to see that

- $z \in \Gamma^{\mathscr{W}^S}(A)$ if and only if $z \in \Gamma^{\mathscr{W}^S_{\alpha,\beta}}(A)$ for all $0 < \alpha < \beta < \infty$ and
- the condition

$$z \in \bigcup_{i \in S, j \in \overline{S}} \mathscr{V}^S_{ij}(A) \cup \Gamma^S_i(A) \cup \Gamma^{\overline{S}}_j(A) \cup V^\alpha_i \cup W^\beta_j \quad \text{for all } 0 < \alpha < \beta < \infty$$

is equivalent with the following one

$$z \in \bigcup_{i \in S, j \in \overline{S}} \mathscr{V}^S_{ij}(A) \cup \Gamma^S_i(A) \cup \Gamma^{\overline{S}}_j(A) = C^S(A). \qquad \square$$

4. Comments and remarks

Indeed, for the minimal Geršgorin set with respect to the special subset $\mathscr{W}$ of $\mathscr{R}$ we have obtained the same formula as in [1]. It requires a lot of additional work, comparing to the set $\mathscr{D}(A)$, which is determined from $n(n-1)$ oval-like sets. Namely, as we have already mentioned, the set $\mathscr{C}(A)$ can be expressed in two equivalent forms (one of them given here, and the second one in the paper [1]). In fact, it could be expressed in many different (equivalent) forms, but all of them are determined from $O(n^2 2^n)$ oval-like sets. Evidently, the work involved increases significantly, but nevertheless, it can lead to good approximation of the minimal Geršgorin set (as it was defined by Varga), which remains unattainable. The following example (in fact, the same one as in the paper [1]), illustrates this.

Numerical example. For the matrix

$$G = \begin{bmatrix} 10 & 0 & 3 & 5 \\ 0 & -10 & 2 & 4 \\ 2 & 5 & 20 & 0 \\ 4 & 4 & 0 & -20 \end{bmatrix},$$

the following figure shows the sets $\Gamma(G)$, $\Gamma^{\mathscr{D}}(G)$, $\Gamma^{\mathscr{W}}(G)$, and $\Gamma^{\mathscr{R}}(G)$, respectively, shaded decreasingly. The last one, minimal Geršgorin set $\Gamma^{\mathscr{R}}(G)$, has been plotted as the union of spectrums of 10 000 random matrices from the extended equimodular set $\hat{\Omega}(G)$ (see [3]). See Fig. 1.

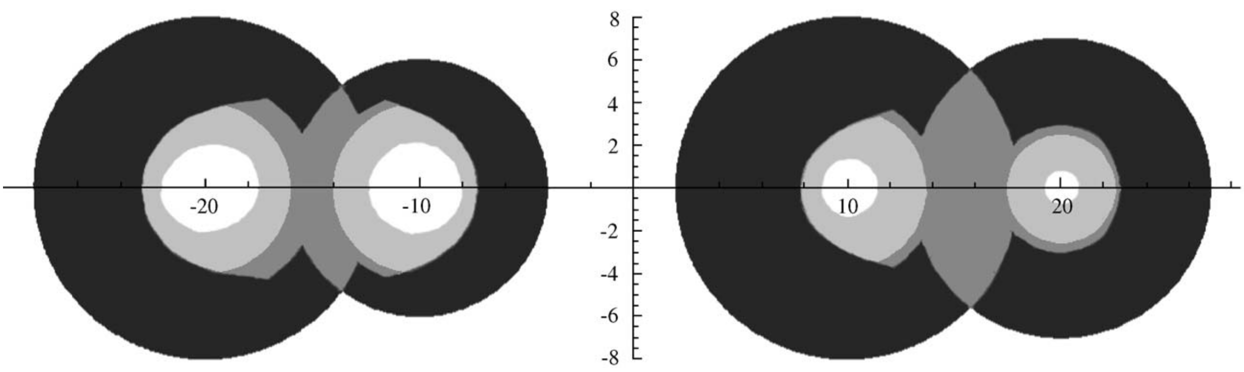

Fig. 1.

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